The KdV Equation and its Soliton Solutions via the Inverse Scattering Transform Method

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This article gives an an introductory account to obtaining solutions, in the form of non-linear, travelling-wave solitons, to the Korteweg de Vries (KdV) equation. These solutions are obtained using the relatively modern method known as the Inverse Scattering Transform, developed in the 1960s.

The mathematical level assumed is that of advanced undergraduate, requiring only a basic knowledge of quantum mechanics.

Contents

Acronyms and Abbreviations

GLM : Gelfand–Levitan–Marchenko
IST : Inverse Scattering Transform
KdV : Korteweg de Vries
SE : Schroedinger Equation (time-independent)

Conventions

Spatial (x) and temporal (t) partial derivatives are given in the common notational form, u_x and u_t respectively, as commonly used in PDE work, i.e.

$$u_t = \partial_t u = \frac{\partial u}{\partial t}$$
$$u_x = \partial_x u = \frac{\partial u}{\partial x}$$
$$u_{xx} = \partial_x^2 u = \frac{\partial^2 u}{\partial^2 x}$$

The dot notation is used for the total derivative with respect to free parameter t, and also for the partial derivative wrt t when there is no confusion, e.g. for some eigenvalue $\lambda = \lambda(t)$ then

$$\dot{\lambda}(t) = \frac{d\lambda}{dt} = \frac{\partial \lambda}{\partial t}$$

Symbols used are as generally per the key source reference Dunajski [1].

The terms *Hermitian* and *self-adjoint* are equivalent and define an operator, e.g. L, in matrix or derivative form, that is the same as its complex conjugate, transpose, i.e. $L^\dagger = L$.

Literature Sources

Several sources of information were used for this article: Dunajski [1]; Tabor [2]; Drazin [3]; Ablowitz [4]. In particular, the text closely follows Section 2 in Dunajski [1]; generally, but not exclusively, adopting the same notation and conventions. Tabor [2] is very readable, as too Drazin [3]. The Ablowitz slide presentation [4] is far more mathematically involved, albeit it does give a good, short historical context.

1 The KDV Equation

The KDV Equation is usually given in most texts in the following, bare-bones form:

$$u_t + 6uu_x - u_{xxx} = 0 \tag{1}$$

The signs of the terms may vary from the above quoted form of KdV, which is from Dunajski [1], whilst Tabor [2] and Drazin [3] use $u_t - 6uu_x + u_{xxx}$, i.e. $u \to -u$, and Ablowitz [4] uses $u_t + 6uu_x + u_{xxx}$ where $u \to -u$ and $t \to -t$.

The above form is a notationally simplified version of the physical PDE describing water waves in a shallow channel, first documented (and solved in part) by Korteweg and de Vries (KdV) in 1895 [5]. Their work was based upon non-linear, solitary waves that were first observed in 1844 by the engineer John Scott Russell [6], who also studied such waves (termed *solitons*

in more recent times) [1] , and gave an equation for the speed of the wave. The unexpected stability of these non-linear waves was controversial at the time, and it wasn't until 1895, when KdV published the first, single-soliton solution [5], that their genuine existence became more acceptable.

General non-linear, but stable physical phenomena, largely remained unstudied for a long time, until stable solutions for other non-linear phenomena started to appear, notably beginning with the work of Fermi, Ulam and Pasta in 1955 [7] on stable, non-dissipative solutions in weakly coupled, non-linear oscillators. Needless to say, many more non-linear PDEs are now known, all with stable solutions, and advanced techniques to solve them, not least the Inverse Scattering Transform (IST) method for KdV, [8] [9], building on earlier work by Gelfand, Levitan and Marchenko [10], [11].

Returning to the KdV equation (1), this has a non-linear term uu_x, plus a cubic dispersive term u_{xxx}, the effect of dispersion is to produce waves with a group velocity that is not equal to its phase velocity, whereby different frequency components travel with different speeds, and hence the wave literally disperses. The non-linear term essentially makes the wave solution non-stable, i.e. the wave breaks up exactly as per an ocean wave breaking as it comes onshore. See [1] for the mathematical background on going from simple linear waves to non-linear solitons via the addition of non-linearity and dispersion.

However, it is evident that the non-linearity acts counter to the dispersion, with the result that stable, non-dispersive solutions exist. A neat feature of this is that two solitons can collide (scatter) and regain their original form (shape), post-scattering, as their separation becomes ever-larger - this was most definitely not expected prior to the modern (last sixty or so years say) study of non-linear phenomena. By way of comparison, electromagnetic radio waves, for example, are linear, non-dispersive (in air at least) and interfere non-destructively - radio communication would not be possible if the waves destructively interfere. This stability being mathematically due to the ability to represent such waves, before and after interfering, by the same Fourier decomposition into a trigonometric series with constant coefficients. Indeed, this linear decomposition can be used to obtain solutions to linear differential equations, but is generally not so, for non-linear phenomena, until the method of the Inverse Scattering Transform (IST) appeared, when analogous solutions for non-linear equations emerged. Hence IST is also known as a non-linear Fourier transform.

2 A simple solution

The KdV can be solved directly in a few simple cases, and does not require the full armoury of the IST. However, this is only for the first few integrals, and not a general solution, nor generic to other non-linear problems. Nevertheless, these solutions provide insight and give solutions that more generic methods must also obtain. This section details the analytic solution for the simplest, single-soliton case, since this is also a textbook first-case for the more-general IST method, detailed later herein. The reader is referred to Tabor [2] and Drazin [3] for a full, readable, account of the first few simple solutions.

Defining the spatial coordinate X by the following *Galilean* transform, for constant phase ve-

[1] A soliton is more generally known as any stable solution to a non-linear PDE that has the form of a non-dispersive, travelling wave that can scatter off other solitons, and both retain their original form post-scattering in the asymptotic limit as they separate to infinity). Their scattering interaction is given by a non-linear potential function. The retention of form post-scattering gives them their particle nature, hence the name soliton as per other particles, e.g. protons, electrons. Needless to say, their are numerous other such *pseudo* particles in physics, notably magnons, excitons, phonons etc., mainly, but not exclusively, the domain of condensed matter physics.

locity v_p:

$$X = x - v_p t \tag{2}$$

and defining the KdV solution $u(x,t)$ in terms of just X (the justification follows shortly) by

$$u = u(X) \tag{3}$$

then the partial derivatives u_x and u_t in the KdV equation (1) are

$$\begin{aligned} u_x &= \frac{\partial X}{\partial x}\frac{d}{dX}u = u_X \\ u_t &= \frac{\partial X}{\partial t}\frac{d}{dX}u = -v_p u_X \end{aligned} \tag{4}$$

Inserting these partial derivatives into KdV gives

$$-v_p u_X + 6uu_X - u_{XXX} = 0 \tag{5}$$

which is now a non-linear ODE (instead of a non-linear PDE) for u in terms of X, and easily re-written as the following total derivative:

$$\frac{d}{dX}\left(-v_p u + 3u^2 - u_{XX}\right) = 0 \tag{6}$$

Integrating and rearranging, for some constant c_1, gives

$$u_{XX} = 3u^2 - v_p u + c_1^2 \tag{7}$$

Re-writing u_{XX} in a familiar, first order derivative form gives

$$\frac{1}{2}\frac{d}{du}u_X^2 = 3u^2 - v_p u - c_1 \tag{8}$$

and thus yet another easy integration can be performed to leave a first order ODE

$$\frac{1}{2}u_X^2 = u^3 - \frac{v_p}{2}u^2 - c_1 u + c_2 \tag{9}$$

The constants c_1 and c_2 can both be set to zero because u and its first (u_X) and second derivative (u_{XX}), can be assumed to be zero at $\pm\infty$ since u represents a potential field with no effect at these limits. In which case, this becomes

$$u_X = u\sqrt{2u - v_p} \tag{10}$$

Rearranging and partially integrating gives, for initial value X_0,

$$\int \frac{du}{u\sqrt{2u - v_p}} = X - X_0 \tag{11}$$

And, lastly, the integral on the left is can be evaluated as standard to give

$$u(X) = -\frac{v_p}{2\cosh^2\left(\frac{1}{2}\sqrt{v_p}(X - X_0)\right)} \tag{12}$$

For comparison with the same *single soliton* solution obtained later (71), the velocity v_p is defined in terms of the eigenvalue χ by

$$v_p = 4\chi^2 \tag{13}$$

and the above solution becomes, in terms of χ,

$$u(X) = -\frac{2\chi^2}{\cosh^2\left(\chi(X - X_0)\right)} \tag{14}$$

In addition, the transformed variable X (2) is thus

$$X = x - 4\chi^2 t + X_0 \tag{15}$$

This is the standard form of the simplest, single soliton solution.

3 Solving KdV via IST

3.1 Overview

Obtaining the solution $u(x,t)$ to the KdV equation is equivalent to solving (albeit indirectly) the time-independent Schroedinger equation (SE) with the same potential $u(x,t)$, where t is not to be particularly regarded as time (hence *time-independent*), but rather as a free parameter with which to vary the potential. In other words, the potential is effectively two-dimensional.

In fact, the potential $u(x,t)$ can be obtained by analytically solving the SE for an initial solution when $t = 0$, and then evolving this eigenvector solution (scattering data) forward for non-zero t by using methods due to Lax [12]. Using the evolved eigenvector solution, the potential $u(x,t)$ is then effectively reconstructed by the Inverse Scattering Transform Method (IST), which is just the desired solution to KdV for all x and t.

The principle here being that it is easier to solve the *indirect* (scattering) problem, i.e. obtain the eigenvector solution to the SE for some non-zero t, without having to analytically solve SE explicitly. Only the SE solution at $t = 0$ is required, thereafter, using some cunning, the SE eigenvector solution can be obtained with just this initial solution. Of course, in practice, it is not so simple and involves, in the general case, solving a complicated integral equation. However, for KdV, simplifications can be made and the result is a general multi-soliton solution.

3.2 Process

The process to obtain a KdV solution is summarised in four acronyms and one name (Lax), given below in sequential order starting with the Schroedinger equation (SE)

- SE
- Lax
- IST
- GLM
- KdV

3.2.1 SE

The first step to a KdV solution is to solve the time-dependent Schroedinger Equation (SE) for a given potential $u(x,t)$, at $t = 0$, i.e. $u(x,0)$, to obtain the initial scattering data (basically the solution), which comprises the following:

Energy eigenvalues
Reflection coefficients (eigenvalue-dependent)
Transmission coefficients (eigenvalue-dependent)

The reflection and transmission coefficients are really just simple functions of the coefficients (a and b herein), as would be obtained as standard when solving SE to obtain a normalised eigenvector solution.

The very fact this reflection/transmission terminology is used implies the potential function is of standard potential well or barrier-type, commonly employed when giving simple solutions to SE in standard undergraduate QM texts, e.g. [13]. Indeed, for the purposes of this article, the potential function is never explicitly stated but important assumptions are made that it and its derivatives vanish at infinity, giving rise to a *bound state* solution with only a finite number of discrete eigenvalues. In fact, only the eigenvector solution, in the asymptotic limit ($x + \pm\infty$), is used herein, but, ultimately, the eigenvalues (in the central, potential region), must be known

since they parametrise the final KdV solution, albeit it is of a single hyperbolic form as given for the simple KdV solution in the previous section.

3.2.2 Lax

The methods developed, due to Lax [12], take the initial SE scattering data for $t = 0$ and propagate this forward for non-zero t, effectively giving the SE solution for any value of t.

It is only at this stage that the explicit KdV PDE 1 enters via a new operator A, that, together with the existing Schroedinger, time-independent operator L (17), further below, effectively combine to give the KdV PDE. Determining this operator A is not obvious, but has been done for KdV and many other similar, non-linear problems.

The two operators, L and A, are known as a *Lax pair* and, together, give some very important properties of the t-evolved solution, notably that the eigenvalues are t-independent, thus the same for the entire solution as those obtained for $t = 0$. This t independence known as the *isospectral* property.

Having obtained the t-evolved scattering data, the full KdV solution $u(x, t)$ is obtained by solving a linear integral equation.

3.2.3 IST and GLM

The Inverse Scattering Transform (IST) [9] is essentially a process that takes the t-evolved scattering data and inputs it to the Gelfand-Levitan-Marchenko (GLM) equation [10], [11], which is then solved to reconstruct the potential $u(x, t)$, i.e. that which gives the same t-evolved scattering data as if the SE equation were solved analytically in the first place, i.e. knowing $u(x, t)$ in advance, but not the scattering data for non-zero t.

Ordinarily, this involves solving a rather fiendish-looking integral equation. However, considerable simplifications, not least assuming a *reflectionless* potential, reduce it to only requiring the t-evolved, bound-state SE solution, i.e. the reflectionless t-scattering data - a case of scattering with no reflection!

3.2.4 KdV

The KdV solution is simply the potential function $u(x, t)$ as output by the IST method. The solution is parametrised by the eigenvalues of the original, $t = 0$, potential $u(x, 0)$

To reiterate a point earlier, the exact form of the potential u is not specified, albeit it has to be known to give the eigenvalue solution for $u(x, 0)$. Providing the potential satisfies its convergence criteria, giving a bound-state, finite number of discrete eigenvalues, the potential can be quite arbitrary and still give the same form of hyperbolic soliton solution, albeit uniquely distinguished by the eigenvalues of the chosen potential.

3.3 The Time-independent Schroedinger Equation

The time-independent Schroedinger equation for eigenstate ψ, energy E, and position-dependent potential $u(x)$ is given in standard form [13] by

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + u(x)\right]\psi = E\psi \tag{16}$$

By defining the *Lax* operator L as follows, which is the same as the time-independent Hamiltonian here (square bracketed expression above),

$$L = -\partial_{xx} + u \tag{17}$$

then (16) is re-written in simpler operator form as

$$L\psi = E\psi, \quad \frac{\hbar^2}{2m} = 1 \tag{18}$$

The potential u is a function of x, but will later be expanded to include an additional degree of freedom via parameter t, whereby t, as regards the Schroedinger equation, is not to be particularly associated with time. This is because only the time-independent form of the Schroedinger equation need be solved for an initial, time-independent potential $u(x) = u(x,t)$ at $t = 0$.

In fact, t is known as a deformation parameter [2], and merely *deforms* the potential by adding an extra degree of freedom with which to vary it, as in $u = u(x,t)$. Note too that, whilst the Schroedinger equation above is simply given as $u(x)$, not $u(x,t)$, by introducing t, the eigenvalue E also now superficially becomes a functions of t, but this t-dependence is later nullified since $dE/dt = 0$, known as the *isospectral* property, Section (3.5).

Nevertheless, the parameter t does take on a temporal meaning when the final KdV solution is obtained as a non-linear wave solution that propagates along the single spatial x-direction with time-dependence, i.e. a unidirectional travelling wave solution.

The potential function must satisfy the key asymptotic condition that it and its first derivative vanish at spatial infinity as in

$$\lim_{x\to\pm\infty} |u(x,t)| = 0, \quad u_x = 0 \tag{19}$$

The full GLM/IST soliton solution for KdV, Section (3.8), actually requires a more constrained potential as given by the following integral:

$$\int_{-\infty}^{\infty} (1 + |x^2|)u(x,t)dx < \infty \tag{20}$$

where the '$< \infty$' condition means finite, not infinite, and restricts the eigenvalues to a finite, discrete set. For now, this extra condition is not explicitly required, but will be enforced later when obtaining soliton solutions.

Given it is known in advance that the potential function $u(x,t)$ is a KdV soliton solution then, for the purposes of relating back to standard QM, one can crudely think of it as a finite height barrier potential, i.e. a rectangular-shape with constant height U_0, width $2a$, centred around the origin (for some fixed value of t), i.e. $u(x) = U_0$ for $-a < x < a$, with $u(x) = 0$ outside this box, i.e. in the two regions $x < -a$ and $x > a$. One can then solve SE for the case of an impinging plain wave, as will be stated shortly, to get some idea of the eigenvalues and eigenfunctions. However, keep in mind this is just a loose example for SE, and not the KdV solution.

3.4 Plane Wave Asymptotic Solution

For an incident wave travelling from right to left [2] , the asymptotic solution to the SE (16), for a potential satisfying (19), is

$$\lim_{x\to-\infty} \psi(x) = e^{-ikx} \tag{21}$$

$$\lim_{x\to\infty} \psi(x) = a(k)e^{-ikx} + b(k)e^{ikx} \tag{22}$$

[2] See [13], which details an opposite, left-to-right travelling wave solution for a barrier potential of finite height V_0 and width. The incident wave has both a plane-wave transmitted component on the left, as $x \to -\infty$, and an incident plus reflected component to the right of the potential barrier as $x \to \infty$. To reverse the travelling wave direction, simply negate the sign of x.

It is seen that this solution is formed from a basis of two plane waves, i.e. $e^{\pm ikx}$, and is only asymmetric in form (first equation wrt second) due to the initial one-way direction of wave travel (right-to-left).

Indeed, for $u = 0$, the Schroedinger equation, in the form (18), reproduced below

$$\frac{d^2}{dx^2}\psi = -k^2\psi, \quad E = k^2, \quad u = 0 \tag{23}$$

is that of a free particle for a stationary state with a general solution

$$\psi = ce^{-ikx} + de^{ikx}, \quad cc^\dagger + dd^\dagger = 1$$

The first solution (21) is the transmitted wave, and the second solution comprises an incident wave component, coefficient $a(k)$, and a reflected component, coefficient $b(k)$. Remember, the incident wave moves from right ($x = \infty$) to left ($x = -\infty$) hence e^{-ikx}, whilst the reflected wave moves from left to right, hence e^{ikx}.

The coefficients a and b in (21) relate to the standard QM [13] reflection r and transmission coefficients t by

$$r(k) = \frac{b(k)}{a(k)}, \quad t(k) = \frac{1}{a(k)} \tag{24}$$

The coefficients a, b, eigenvalues k^2 and eigenfunctions ψ form what is collectively known as the initial *scattering data* for $t = 0$, given in the asymptotic limit $x = \pm\infty$. In fact, the eigenvalues will be seen to be t-independent, known as *isospectral*, but the a, b coefficients are functions of t, i.e. $a(k, t)$, $b(k, t)$, known as the time-evolved scattering data, and to be determined.

Whilst it appears that this is a sinusoidal-type (or cosinusoidal) wave solution, this is because the energy eigenvalue is assumed positive, and k is thus real-valued ($E = k^2$ above). However, it is possible that $E < 0$ and $k \in \mathbb{C}$, in which case pure exponential solutions arise, e.g. $k = i\chi, \chi \in \mathbb{R}$, $e^{\pm ik} = e^{\mp\chi}$, and it is this latter case that is actually of most relevance in the soliton solution. However, it is not necessary to make this distinction as yet, and the theory will be developed for general k, real or complex.

Thus, so far, only an asymptotic solution for an arbitrary potential, albeit one that vanishes at $\pm\infty$, has been given. The next stage is to get the t-evolved scattering data, and to do this requires a method due to Lax [12], discussed next.

3.5 Lax Pair

A Lax pair comprises two operators L, A, that satisfy the following commutator relation, known as the *Lax equation*, where $\dot{L}$ is the time (or just parameter t derivative) of L:

$$\dot{L} = [L, A] \tag{25}$$

In general, operator L is a linear polynomial in d/dx i.e.

$$L = \sum \frac{d^n}{dx^n} \tag{26}$$

and, as for a standard linear ODE, a function f is an eigenfunction solution [3] , eigenvalue E, to this operator, i.e.

$$Lf = Ef \tag{27}$$

[3]The switch from eigenfunction ψ, used in the earlier SE solution, and f here, was done only to match the text in Dunajski [1]. ψ generally gets used in QM texts, but the use of f here is generic, and the Lax methods are not restricted to QM and/or SE.

In fact, L will be equated with the Schroedinger operator (17), but this explicit form, nor that of A, are required for this general discussion on Lax pairs. They (L and A) will, however, be required in the next section when solving for the evolution of scattering data.

This and the Lax equation (25) are almost sufficient to demonstrate the key *isospectral* property of L, namely that eigenvalue E is t-parameter independent. This t independence is very important, since it means that the eigenvalue is a constant of the system, i.e. a *first-integral*. When t is a temporal parameter then E is time-independent, i.e. it does not change with time, and thus a conserved quantity. For KdV, the eigenvalues (later denoted by $\chi_n, n = 1 \dots N$ for an N-soliton solution) are isospectral and thus constants of motion. Each eigenvalue has an associated eigenfunction, and thus a first integral of motion.

The final condition is that L is self-adjoint, which also implies E is real-valued, i.e.

$$L^\dagger = L \Rightarrow E \in \mathbb{R} \Rightarrow E^\dagger = E \tag{28}$$

This self-adjoint property of L is a staple of all operators that represent an observable in quantum mechanics, because it ensures real-valued, positive eigenvalues and, consequently, real-valued observables with non-negative probabilities ([13]).

Note that the operators L, A herein are given in differential form, but they can also be given in matrix operator form.

Now to the *isospectral* property. Differentiating the eigenvector equation (27) with respect to t gives, in full,

$$\dot{L}f + L\dot{f} = \dot{E}f + E\dot{f} \tag{29}$$

Using the Lax equation (25) for $\dot{L}$, and expanding, this becomes

$$LAf - ALf + L\dot{f} = \dot{E}f + E\dot{f} \tag{30}$$

and using the eigenvector equation (27) a second time for the Lf term, and also using $EA = AE$ for scalar eigenvalue E, gives

$$LAf - EAf + L\dot{f} = \dot{E}f + E\dot{f} \tag{31}$$

This then factors nicely resulting in

$$(L - E)(\dot{f} + Af) = \dot{E}f \tag{32}$$

Taking the inner product of this expression with $f^\dagger$, and again using $E\dot{A} = \dot{A}E$ for scalar eigenvalue E, then

$$f^\dagger(L - E)(\dot{f} + Af) = \dot{E}f^\dagger f = \dot{E} \parallel f \parallel^2 \tag{33}$$

In accordance with (28), the term $L - E$ itself is self-adjoint, i.e.

$$(L - E)^\dagger = (L - E) \tag{34}$$

and using this, then the term $f^\dagger(L - E)$ can be re-written as follows:

$$f^\dagger(L - E) = [(L - E)^\dagger f]^\dagger = [(L - E)f]^\dagger \tag{35}$$

and given $(L - E)f = 0$ by definition (27), then the left term in (33) is zero, and this therefore implies

$$\dot{E} \parallel f \parallel^2 = 0 \tag{36}$$

The first isospectral condition

Lastly, since $\| f \|^2$ is non-trivially always positive definite, the final conclusion is that, given L is self-adjoint, then the derivative of the eigenvalue E wrt t is zero, i.e.

$$L^\dagger = L \Rightarrow \| f \|^2 > 0 \Rightarrow \dot{E} = 0 \tag{37}$$

This property is known as the the *isospectral* property of the eigenvalues of L. Of course, it is not just the fact that L is self-adjoint that gives the isospectral property, but also that it is part of a Lax pair L, A, satisfying the commutation relation (25) for some, as yet unspecified, matrix A.

It is also important to note that L is a function of parameter t, i.e. $L = L(t)$, via (17) and the fact that $u = u(x,t)$, albeit the t-dependence of u is not explicitly shown. Indeed, $\dot{L}$ is given by the non-trivial commutation relation (25), used to obtain this isospectral result. So too, of course, are the eigenvectors (f) functions of t, i.e. $f = f(t)$, so the isospectral property amounts to a system whereby the operators and their eigenvectors evolve with t, whilst the eigenvalues do not.

A second isospectral condition

Obtaining isospectral eigenvalues for L does not necessarily mean L has to be self-adjoint, and there is, in fact, another way to achieve time-invariant eigenvalues by specifying instead that the matrix A relates the eigenvectors to their time-derivatives by

$$Af = -\dot{f} \tag{38}$$

This is the approach taken by some authors (to within a sign of A), e.g. Ablowitz [4], see further.

Using this property of A, then it is seen that the bracketed term $\dot{f} + Af$, on the left in (32), is also zero, i.e.

$$Af + \dot{f} \Rightarrow \dot{E}f = 0 \Rightarrow \dot{E} = 0 \tag{39}$$

and, again, if eigenvector f (of L) is non-trivially zero (which it isn't - eigenvectors are never zero except trivially), then the conclusion is that E is isospectral.

To summarise, isospectral eigenvalues can be achieved either by asserting L is self-adjoint, or by asserting the matrix A relates eigenvectors f to their derivatives wrt t. Either condition is evidently sufficient.

Returning to (32), for $\dot{E} = 0$, then

$$(L - E)(\dot{f} + Af) = 0, \quad \dot{E} = 0 \tag{40}$$

and, if it assumed that L is self-adjoint, but $(\dot{f} + Af) \neq 0$, then this equation simply rearranges to

$$L(\dot{f} + Af) = E(\dot{f} + Af) \tag{41}$$

and so it is seen that if f is an eigenvector of L, then so too is $(\dot{f} + Af)$ for the same eigenvalue E. This means the eigenstate E is degenerate, with two eigenvectors f and $\dot{f} + Af$. Of course, if L is not self-adjoint, but the other condition (38) holds, then the quantity $(\dot{f} + Af)$ is trivially zero and not therefore an eigenvector.

To summarise the above

Given an operator L, with eigenvectors f and eigenvalues E (27), and some matrix A, the two L, A form what is known as a Lax pair (25) defined as follows, reproduced from earlier:

$$\dot{L} = [L, A]$$
$$Lf = Ef$$

If L is self-adjoint, the eigenvalues are constant with respect to parameter t, a condition known as *isospectral*

$$L^\dagger = L \Rightarrow \dot{E} = 0 \tag{42}$$

Alternatively, if the matrix A relates the derivatives of the eigenvectors wrt t by (38), then the isospectral condition is also met regardless of whether L is or is not self-adjoint, i.e.

$$Af = -\dot{f} \Rightarrow \dot{E} = 0 \tag{43}$$

If the second isospectral condition (38) is enforced then, since this is a simple and familiar first order differential equation, the eigenvectors f are seen to evolve parametrically with t as follows, where f_0 is the initial eigenvector at $t = 0$

$$f(t) = f_0 e^{-At} \tag{44}$$

This parametric evolution is discussed again shortly when comparing with QM in the below notes. Suffice to say, isospectral eigenvalues are constant by definition, and hence first integrals.

The Lax operator L (17) is that of the Schroedinger equation (16) for a general potential $u(x, t)$ where t is to be thought of as any arbitrary parameter, known as a *deformation* parameter. Operator L actually has no explicit connection to KdV, rather this connection comes with the other operator A in the Lax pair, which has not yet been defined, see the next section (3.6).

The key leap to KdV comes about by defining A, whereby the Lax equation (25) then effectively becomes KdV (1). The importance of A is due to the fact that not only is f an eigenfunction of L, but so too is $\dot{f} + Af$. In other words, given f, which can be taken to be an initial solution to the Schroedinger equation for potential u at $t = 0$, then another t-evolved eigenvector can be obtained by solving $\dot{f} + Af$ as a first order differential equation in f. However, since A can be quite involved (and non-linear), this may not seem so easy. Nevertheless, as for much of this KdV solution, it is about asymptotics, i.e. knowing the solution at $x = \pm\infty$ (note, not $t = \pm\infty$). In particular the non-linearity of A disappears at this limit. This fact will be used later when evolving the scattering data, Section (3.6).

Lastly, to close this section, the Lax pair L, A are only strictly unique to within a similarity transform, see Appendix (6.2).

3.6 Scattering Data Evolution

Having detailed, in the previous section, how t-dependent evolution of eigenvector states f can be achieved via Lax matrix A (43), it is now shown how to use A in combination with the asymptotic state solution (21), reproduced below, to obtain t-evolved scattering data.

$$\lim_{x\to-\infty} \psi(x) = e^{-ikx}$$
$$\lim_{x\to\infty} \psi(x) = a(k)e^{-ikx} + b(k)e^{ikx}$$

Firstly, from (41), if f (now replaces ψ, above) is an eigenvector of L, then so too is $\dot{f} + Af$. As noted earlier, matrix A is one-half of the Lax pair, and is defined as

$$A = 4\frac{d^3}{dx^3} - 3\left(u\frac{d}{dx} + \frac{d}{dx}u\right) \tag{45}$$

As a reminder, L is explicitly defined as the Schroedinger operator (17), and relates to A via the Lax equation (25), both reproduced below for ease of reference

$$L = -\partial_{xx} + u$$
$$\dot{L} = [L, A]$$

This is the first time that A has been defined but, given L, it can be seen, with a bit of straightforward algebra, that L and A satisfy the Lax equation - hint evaluate $[L, A]v$ for some arbitrary function v and note that, using KdV (1), it is equal to $\dot{L}$ - the relation $\dot{L} = \dot{u}$ is also required, see the proof in Appendix (6.1).

As one can see, A does not have the simplicity of L (17) but, then again, neither is the KdV equation as simple as the time-independent Schroedinger equation (16), and it is A that effectively links the two - somewhere the non-linearity of KdV has to appear, and A is that place!

Given u and its first derivative u_x vanish at infinity, the asymptotic form of A is

$$\lim_{x\to\pm\infty} A = 4\frac{d^3}{dx^3} \tag{46}$$

and thus its asymptotic operation on f is given by

$$\lim_{x\to-\infty} Af = 4ik^3 f \tag{47}$$

Reminder. Eigenfunction f is just a relabel of ψ, which is an eigenvector solution of the Schroedinger equation (18), i.e. $Lf = Ef, E = k^2$.

From earlier, it was seen that if f is an eigenvector of L, then so too is $\dot{f} + Af$ with the same asymptotics, and so

$$\lim_{x\to-\infty} \dot{f} + Af = 4ik^3 f \tag{48}$$

It is argued, Dunajski [1], that, whilst this is only given in the asymptotic region $x \to \pm\infty$, it actually holds true for all x. A more rigorous, but rather detailed, algebraic approach to obtaining the same result above is given by Tabor [2], Drazin [3] and Ablowitz [4]. The important point though is that this means f satisfies the following first order differential equation for all x, i.e.

$$\dot{f} + Af = 4ik^3 f, \quad -\infty < x < +\infty \tag{49}$$

Indeed, the second asymptotic solution (21), for $x \to \infty$, can now be substituted into the above to give the following differential equation in the scattering coefficients a, b:

$$\dot{a}e^{-ikx} + \dot{b}e^{ikx} = 8ik^3 be^{ikx} \tag{50}$$

Equating the exponentials gives

$$\dot{a} = 0 \quad \dot{b} = 8ik^3 b \tag{51}$$

and so, upon integration, the t evolution of the scattering coefficients is

$$\Rightarrow a(k,t) = a(k,0), \quad b(k,t) = b(k,0)e^{8ik^3 t} \tag{52}$$

Noting that a has no parametric t-dependence, i.e. it does not evolve with t.

3.7 Bound states

As mentioned earlier in Section (3.4), there has been no assumption on the *isospectral* eigenvalues E, as regards whether they form a continuous or discrete spectrum, and whether the related wave-numbers k are real or complex, $E = k^2$ (23).

In fact, the further GLM development below covers both cases, but the case of interest is the bound state where the energy eigenvalues E are less than zero and comprise a discrete, finite set. Since E is negative, then the wave-number k is now complex ($k^2 = E, E < 0$) and re-written in terms of a real-valued variable χ [4] , as in:

$$E < 0, \quad k^2 = E \Rightarrow k = i\chi, \quad \chi \in \mathbb{R} \tag{53}$$

Such a discrete eigenvalue spectrum was alluded to in the second condition (20) on the potential $u(x,t)$. In fact, as in general QM, the asymptotic solution is required to be *square integrable*, i.e.

$$\int_{-\infty}^{\infty} |\psi|^2 dx < \infty \tag{54}$$

When normalised this would be unity. However, for a plane wave (e.g. Ae^{ikx}, whose amplitude A remains constant for all $-\infty < x < \infty$), this is not the case. Indeed, the plane wave solution has a continuous eigenspectrum ($k^2 \in \mathbb{R}$) and so has to be reconsidered in the bound state case.

Furthermore, with the number of discrete eigenvalues finite (denoted by N), each eigenvalue, associated eigenfunction, and all other related quantities, notably the scattering coefficients a and b, are explicitly subscripted n, where $n = 1 \ldots N$:

$$\begin{aligned} \psi \to \psi_n, \quad k \to i\chi_n, , \quad n = 1 \ldots N \\ a(k) \to a(i\chi_n) \sim a_n, \quad b(k) \to b(i\chi_n) \sim b_n \end{aligned} \tag{55}$$

With this in mind, the general, asymptotic solution (21) has to be revisited in the negative energy, discrete, bound state case, and is reproduced below using the above notational changes: Note that the original solution actually remains perfectly valid in the continuous case and will not be altered.

$$\begin{aligned} \lim_{x \to -\infty} \psi_n(x) = e^{\chi_n x} = 0 \\ \lim_{x \to \infty} \psi(x) = a_n e^{\chi_n x} + b_n e^{-\chi_n x} = \infty \end{aligned} \tag{56}$$

Applying the limit to the first of these shows that it is zero, whereas the first term in second equation is infinite. As such, this term is removed, leaving the bound state, discrete solution rewritten as follows:

$$\begin{aligned} \lim_{x \to -\infty} \psi_n(x) = e^{\chi_n x} \\ \lim_{x \to \infty} \psi_n(x) = b_n e^{-\chi_n x} \end{aligned} \tag{57}$$

This solution is not that of an incident plane wave, and it is actually a bit nonsensical to speak of it in terms of reflection and transmission coefficients. If one remembers their QM, with a particle of energy $-E$ sitting in a square well potential of finite energy depth $-U_0$ and finite width, where $E, U_0 > 0$ and $|U_0| > |E|$, so the particle cannot escape, i.e. it is bound, then

[4] Use of χ is Dunajski notation [1]; others, e.g. Drazin [3] and Ablowitz [4], use κ, and Tabor [2] keeps with k - beware!

outside the well the particle's wave-function decays exponentially to zero in the limit $x = \pm\infty$ - essentially that is the above asymptotic solution. To get the full, stationary, bound state eigenvalues, one has to solve the SE for the potential in the well, and match the wave-function and its derivatives, on the well boundary, with the wave-function solution outside the well - all standard QM, see [13].

The above will thus give the initial, $t = 0$ solution (for some initial potential $u = u(x, t), t = 0$ - not necessarily a square-well) and, exactly as before in the continuous case, the coefficients thereafter evolve with t. The b coefficient has already been given earlier (52), but is reproduced below in its explicit, discrete form, as too a now asserted as zero for all t:

$$\begin{aligned} b_n(i\chi_n, t) &= b_n(i\chi_n, 0)e^{8\chi_n^3 t} \\ a_n(i\chi_n, t) &= a_n(i\chi_n, 0) = 0 \end{aligned} \tag{58}$$

Aside. Summary of discrete and continuous QM states

Continuous. Unbounded state, positive energy, real-valued eigenvalues and wave numbers, complex exponential eigenfunctions (plane wave solutions), eigenfunctions remain finite at infinity, not square integrable.

Discrete. Bounded state, negative energy, complex wave number, real exponential eigenfunctions, finite number of eigenvalues for finite, negative energy, i.e. not an infinite depth potential well, eigenfunctions decay to zero at infinity $e^{kx}x \to -\infty$, $e^{-kx}x \to -\infty$, square integrable.

3.8 Inverse Scattering Transform

Thus, having obtained the t-evolved scattering data $a(k, t)$, $b(k, t)$ (continuous) and $a_n(\chi_n, t)$ (discrete), plus isospectral eigenvalues $E = k^2$ (continuous) and $-E_n = (i\chi_n)^2$ (discrete), the next and final stage in solving KdV entails taking this evolved scattering data and applying the IST to retrieve the potential function $u(x, t)$, which is then the KdV solution.

The Gelfand–Levitan–Marchenko theory (GLM), [10] and [11], shows that, given the scattering data for any t, as just obtained in the previous section, then the potential $u(x, t)$ can be determined from it. This potential being a KdV solution in accordance with the kdV-specific Lax operator A (45) used to t-evolve the scattering data.

Firstly, a normalisation constant $\beta_n(t)$ is defined as follows

$$\beta_n(t) = \left[\int_{-\infty}^{\infty} |\psi_n(x, t)|^2 dx\right]^{-1} \tag{59}$$

where ψ_n is the discrete eigenfunction solution (57) given in the previous section. This 'constant' is a normalisation factor, and is thus indirectly calculated in terms of the discrete scattering coefficient b_n (a_n is zero) given in (57). Because it is a function of ψ, which is implicitly a function of t and t-evolved coefficient b_n (58), β_n is also a function of t. In fact, its t-dependence is given in Dunajski [1] as

$$\beta_n(t) = \beta_n(0)e^{8\chi_n^3 t} \tag{60}$$

which is the same as the t-evolution of b_n (58). This is expected because Dunajski (see the below notational aside) defines it directly in terms of b as in the form $\beta_n = b_n/(ia')$ where a' is a t-independent constant, so that the β_n dependence on t will thus be identical to that of b_n.

Notational aside. The symbol β is retained as per Dunajski [1], but other texts tend to use c in some form, e.g. c_n^2 in [2], c_n [3]. The point is made merely to warn readers that Dunajski uses the definition $\beta(t) = \frac{b_n(t)}{ia'(i\chi_n)}$ where a' does not appear defined anywhere, and similarly, Ablowitz [4] gives the exact same definition, but symbol β is replaced by C_j. Perhaps the a' is a derivative as in da/dt, but discrete coefficient a_n is zero, and the continuous coefficient a has no place in the discrete part (summation with β_n) of $F(x)$, below. It may be that $ia'(i\chi_n)$ is simply the continuous coefficient a with the continuous eigenvalue k replaced by its discrete counterpart $i\chi$. Some support for this notion comes from the fact that the integral term (continuous part) on the right of $F(x)$ uses the reflection coefficient $r(k)$ which is $b(k)/a(k)$ herein, whereas Tabor et al do not define their reflection and transmission coefficients normalised by $1/a(k)$. Anyway, be warned, but β, as defined above, is correct and should match the Dunajski definition of β whatever the definition of a' happens to be!

Using the definition of β_n, a function $F(x)$ is defined as follows:

$$F(x) = \sum_{n=1}^{N} \beta_n e^{-\chi_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk \tag{61}$$

and, using this function, the following *GLM integral equation* is to be solved for function $K(x, y)$:

$$K(x, y) + F(x + y) + \int_{x}^{\infty} K(x, z) F(z + y) dz = 0 \tag{62}$$

The potential $u(x)$ in the Schroedinger equation (16), is then given in terms of K by

$$u(x) = -2 \frac{d}{dx} K(x, x) \tag{63}$$

As already alluded to in the above *Notational aside*, the function $F(x)$ (61) actually comprises a discrete component, i.e. the summation term involving β_n, as derived from discrete solution ψ_n, and a continuous part, i.e. the integral containing the continuous reflection coefficient $r(k)$ (24), as derived from the t-evolved scattering coefficients $a(k)$ and $b(k, t)$ (52) for the continuous solution (22). Note too that the transmission coefficient $t(k)$ (24) is not used in the above.

The function $F(x)$ has a second term that is a evidently a Fourier transform of the reflection coefficients $r(k)$ and, because there the solution is actually reflectionless, this integral is zero and $F(x)$ reduces to a series sum over N eigenvalues χ_n, where n is the number of solitons.

Note that the potential $u(x)$ has no explicit t-dependence, but it is implicit in the β_n and $r(k)$ coefficients - see the earlier comment on β's t-dependence. In other words, this makes $u(x, t)$ a t-dependent potential.

3.9 The Single Soliton Solution

In the single soliton $N = 1$ case, $F(x)$ becomes

$$F(x, t) = \beta(t) e^{-\chi x}, \quad \chi \equiv \chi_n, \ \beta \equiv \beta(t) \equiv \beta_n(t), n = 1 \tag{64}$$

and applying this to the GLM equation (62) gives

$$K(x, y) + \beta e^{-\chi x + y} + \int_{x}^{\infty} k(x, z) \beta e^{-\chi(y+z)} dz \tag{65}$$

where the explicit t-dependence in β (60) has been suppressed purely on the ground of notational brevity - it is implicit.

The above integral equation can be solved by using a solution of the form

$$K(x,y) = K(x)e^{-\chi y} \tag{66}$$

The reason to split it like this into two factor $K(x)$ and $e^{-\chi y}$ is that the latter factor then just cancels throughout to give

$$K(x) + \beta e^{-\chi x} + K(x)\beta \int_x^\infty e^{-2\chi z} dz \tag{67}$$

The integral on the right is easily performed and, after rearranging, gives $K(x)$ as

$$K(x) = -\frac{\beta e^{-\chi x}}{1 + (\beta/2\chi)e^{-2\chi x}} \tag{68}$$

and substituting back into $K(x,y)$ (66) gives

$$K(x) = -\frac{\beta e^{-\chi(x+y)}}{1 + (\beta/2\chi)e^{-2\chi x}} \tag{69}$$

It now remains to determine the t-dependent potential $u(x,t)$ as given by (70), reproduced below

$$u(x) = -2\frac{d}{dx}K(x,x) = -\frac{4\beta e^{-\chi 2x}}{[1 + (\beta/2\chi)e^{-2\chi x}]^2} \tag{70}$$

This is integrated to give the single soliton solution as

$$\begin{aligned} &u(x,t) = -\frac{2\chi^2}{\cosh^2 \chi(x - 4\chi^2 t - \phi_0)} \\ &u,\ u_x \to 0,\ |x| \to \infty \end{aligned} \tag{71}$$

where constant phase ϕ_0 is

$$\phi_0 = \frac{1}{2\chi} log\left(\frac{\beta_0}{2\chi}\right), \quad \beta_0 = \beta(t=0) \tag{72}$$

The explicit appearance of parameter t in the solution, which may now be safely associated with time, comes from the substitution for $\beta(t)$ (60), and hence also the appearance of β_0 in the phase term ϕ_0.

As a reminder, the reason to subdue associating t with time was to avoid any misinterpretation that the original Schroedinger equation was time-dependent. It is t-dependent because t in the SE case acts as a deformation parameter on the potential $u(x,t)$, which is also the same $u(x,t$ as in the above solution. As a deformation parameter, it basically allows for extra parametrisation of the potential forms that can be handled by this entire IST method. In short, t is a deformation parameter in SE, and time in KdV.

Looking at the denominator term in (70), the phase ϕ is given by

$$\phi = x - 4\chi^2 t - \phi_0 \tag{73}$$

and so the point of constant phase $\dot{\phi} = 0$ implies a phase velocity v_p given by

$$\dot{\phi} = 0 \Rightarrow v_p = 4\chi^2, \quad v_p = \dot{x} \tag{74}$$

Since eigenvalues χ_n are real-valued (53), ($\chi = \chi_1$ for this $n = 1$ soliton solution), then $v_p > 0$ and so the wave is travelling to the right. Note too that the amplitude u is unconditionally negative, albeit largely superfluous.

Most importantly, because v_p is a constant, since eigenvalue χ is constant (*isospectral*), the phase velocity is constant and the soliton is therefore non-dispersive, i.e. stable, retaining its shape forever. Admittedly, this is a single soliton solution, and it has not therefore scattered off a second soliton - the two-soliton solution is studied in Dunajski [1]. This account also demonstrates that two solitons can pass through each other and emerge with the same shape - strictly speaking only exactly the same shape in the asymptotic limit. There is actually a phase change during the scattering, albeit such phase changes are largely immaterial.

Given the numerator is proportional to the square of χ, then the soliton amplitude is proportional to the phase velocity, i.e. $\chi^2 = v_p/2$. In fact, since cosh > 1 for all arguments, we see $|u| \leq v_p/2$, i.e. the maximum absolute amplitude is half the phase velocity. The squared denominator also means that u is an even function, i.e. the soliton is symmetric about its max peak.

Collating the above points, the above single soliton solution is finally re-written as

$$u(x,t) = -\frac{v_p/2}{\cosh^2(\chi\phi)}, \quad \phi = (x - v_p t) - \phi_0, \quad v_p = 4\chi^2 \tag{75}$$

4 Summary and Conclusions

The article starts with a statement of the common form of KdV equation, together with a very brief account of its history (over 150 years) in the context of non-linear wave phenomena and soliton solutions. A standard, direct solution is then given for simplest, single soliton case, with which to compare with that to be obtained via the indirect method of Inverse Scattering Transform, which then follows.

The method of Inverse Scattering Transform starts with an initial solution to the time-independent Schroedinger equation for a relatively general potential, albeit one that is constrained to vanishing at infinity, which is an asymptotic condition that features in making simplifying assumptions. The potential is a 2D function of both position, as standard, plus a second deformation parameter t that, ultimately, plays the role of time in the KdV solution. Indeed, the *initial* solution is so-named since t is set to zero to obtain it, but thereafter assumed non-zero. However, t is not to be interpreted as time given that the Schroedinger equation is time-independent, and it merely acts as an additional degree of freedom when specifying the potential.

A stronger constraint on the potential, limiting the solution to bound state solutions with a finite number of discrete eigenvalues, is also assumed later, permitting more simplification to be made in the Inverse Scattering Transform.

The initial ($t = 0$) solution comprises the direct scattering data, namely eigenvalues and reflection and transmission coefficients. This solution is then evolved with respect to t, analogous to time-evolution, using modern methods involving a *Lax pair* of operators, one of which is the original Schroedinger operator L, and the second operator A linking to KdV. Together, this pair of operators show that the eigenvalues in the evolved solution do not actually evolve, i.e. they are independent of t, which is an important simplifying property known as *isospectral*. Of course, the eigenfunctions themselves will evolve, subject to the asymptotic condition, and this will evolve the scattering data since these coefficients relate to normalisation and continuity properties of the eigenvectors (or eigenfunctions).

Finally, the t-evolved solution is input to the method of Inverse Scattering Transform, to obtain a new t-evolved potential function that, when used as a starting potential in the Schroedinger

equation, would give the same t-evolved eigenvector solution. Suffice to mention that the second condition on the potential is imposed, limiting the solution to bound states, and leading to a reflectionless solution that considerably simplifies the solution of an integral equation as part of the Inverse Scattering Transform. In other words, a direct solution of the Schroedinger equation for a 2D (x and t) potential function is obtained indirectly from the t-evolved scattering data of initial solution. Of course, the aim is to solve KdV, not the Schroedinger equation, but because the second Lax operator A relates it all to KdV, the final potential function is, indeed, a solution to KdV.

The KdV solution obtained is a general N-*soliton* solution, where N corresponds to the finite number of eigenvalues. In the simplest, single eigenvalue ($N = 1$)m single soliton case, it is then confirmed the same as that obtained by direct solution at the beginning of the article.

The Inverse Scattering Transform is a general method of solving non-linear PDEs, one of which is the KdV equation, and soliton solutions to KdV can be obtained from it for a 2D potential, subject to a few constraints that considerably simplify the method, but still give a solution applicable to multiple solitons cases under various potentials.

5 References

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6 Appendices

6.1 Appendix 1

Proof $L_t = u_t$

Firstly, for any operator L acting on a vector v, as in Lv, the partial time-derivative $\partial(Lv)/\partial t$ is simply given by the usual calculus product rule

$$(Lv)_t = L_t v + Lv_t \tag{76}$$

From the definition of the time-independent Schroedinger operator L (17), reproduced below

$$L = -\partial_{xx} + u$$

then

$$Lv = -v_{xx} + uv \tag{77}$$

Taking the partial time derivative thus gives

$$(Lv)_t = -(v_{xx})_t + (uv)_t \tag{78}$$

Expanding out the second derivative term on the right, again using the calculus product rule, gives

$$(Lv)_t = (-v_{xx})_t + u_t v + uv_t \tag{79}$$

and equating this with (76) implies

$$L_t v + Lv_t = (-v_{xx})_t + u_t v + uv_t \tag{80}$$

The first term on the right $(-v_{xx})_t$, by the property of both partial derivatives and purely notational reasons, can be written as follows:

$$(-v_{xx})_t = -v_{xxt} = -v_{txx} = -(v_t)xx = -\partial_{xx} v_t \tag{81}$$

and thus (80) rearranges to give

$$L_t v + (L + \partial_{xx})v_t = u_t v + uv_t \tag{82}$$

and since $L + \partial_{xx} = u$, by the definition of L (17), then the $(L + \partial_{xx})v_t = uv_t$ which cancels with that on the far right to give

$$L_t v = u_t v \tag{83}$$

i.e.

$$(L_t - u_t)v = 0 \tag{84}$$

and for arbitrary vector v this implies

$$L_t = u_t \tag{85}$$

hence proven.

6.2 Appendix 2

Transformation of operators L, A

The commutation relation (25), i.e. the Lax equation, is satisfied for any similarity transform on L and A. For example, if g is an invertible operator [5] , then the following transformations:

$$\begin{aligned} L' &= gLg^{-1} \\ A' &= gAg^{-1} \\ \dot{A}' &= g\dot{A}g^{-1} \end{aligned} \tag{86}$$

leave the commutation relations invariant, i.e.

$$\dot{L}' = [L', A'] = g[L, A]g^{-1} = g\dot{L}g^{-1} = \dot{L}' \tag{87}$$

Note that the eigenvectors transform as follows:

$$f' = gf \tag{88}$$

leaving the eigenvalues invariant to a similarity transform, shown in full by

$$\begin{aligned} & L'f' = E'f' \\ & \Rightarrow gLg^{-1}gf = E'gf \\ & \Rightarrow gLf = gE'f \\ & \Rightarrow g(Lf - E'f) = 0 \\ & \Rightarrow Lf = E'f \\ & \Rightarrow E' = E \quad \text{by (27)} \end{aligned} \tag{89}$$

The eigenvalues thus retain their isospectral property under a similarity transform. Furthermore, the above result (87) shows that if L and A form a Lax pair then so too do L' and A'.

[5] Invariably, in the quantum world, g is a unitary operator derived from a Lie-group generator h, e.g. $g(\theta) = e^{i\theta h}$, for some parameter θ.