Unity Root Matrix Theory
Physics in Integers
Overview

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Acronyms Used

DCE = Dynamical Conservation Equation
URM = Unity Root Matrix
URMT = Unity Root Matrix Theory
URM3 = URMT 3x3 Matrix formulation

References

[1] Unity Root Matrix Theory, Physics in Integers. R J Miller
http://www.fast-print.net/bookshop/823/unity-root-matrix-theory-physics-in-integers

[2] Pythagorean Triples as Eigenvectors and Related Invariants, www.urmt.org. R J Miller, 2010.

This presentation is an overview of the six papers, published in [1]. The second of these papers is also available in its entirety in PDF format, see [2].

Unity Root Matrix Theory (URMT) is the study of a special type of integer matrix **A**, which arises from the application of a physical, invariance principle to an abstract 'dynamical conservation equation' (DCE), likened to that of energy conservation.

The Unity Root Matrix

$$\mathbf{A} = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix},$$

The matrix elements are the dynamical variables P, Q, R

$$P, Q, R \in \mathbb{Z},\ (P, Q, R) \neq (0,0,0)$$

and their conjugate forms $\overline{P}, \overline{Q}, \overline{R}$

$$\overline{P}, \overline{Q}, \overline{R} \in \mathbb{Z},\ (\overline{P}, \overline{Q}, \overline{R}) \neq (0,0,0)$$

The dynamical variables have unity root properties, e.g. $P\overline{P} \equiv 1 (\mathrm{mod}\, x)$ and the product $P\overline{P}$, for example, is akin to the squared modulus of a complex number, as in $|Z|^2 = ZZ^*$.

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1 Postulates of URMT

Since the mathematics of URMT is thought to have strong links to Physics in Integers, here are some tentative postulates upon which the theory is founded.

1. The Dynamical Conservation Equation (DCE)

There exists a set of dynamical variables $P,Q,R \in \mathbb{Z},\ (P,Q,R) \neq (0,0,0)$ and their conjugates $\overline{P},\overline{Q},\overline{R} \in \mathbb{Z},\ (\overline{P},\overline{Q},\overline{R}) \neq (0,0,0)$, that satisfy the following dynamical conservation equation for some constant $C \in \mathbb{Z}$.

$$+C^2 = P\overline{P} + Q\overline{Q} + R\overline{R} + (PQR + \overline{P}\,\overline{Q}\,\overline{R})/C$$

2.1 Invariance Principle

There exists a set of coordinates $x,y,z \in \mathbb{Z},\ (x,y,z) \neq (0,0,0)$ and their conjugates $\overline{x},\overline{y},\overline{z} \in \mathbb{Z},\ (\overline{x},\overline{y},\overline{z}) \neq (0,0,0)$ such that the following invariance principle holds

The dynamical equations and their solutions are invariant to a coordinate translation in the dynamical variables.

2.2 Invariance Transformations

The coordinate translation in the dynamical variables is defined as the following transformation, given in terms of the standard coordinates x,y,z, for three, arbitrary integer variations η,δ,ε

$$P \to P + \delta x,\ \overline{P} \to \overline{P} - \varepsilon x$$
$$Q \to Q + \varepsilon y,\ \overline{Q} \to \overline{Q} - \eta y$$
$$R \to R + \eta z,\ \overline{R} \to \overline{R} - \delta z$$

3 Observables

All observables in URMT are functions of sums, or functions of sums of products, of one or more variables, with their conjugate forms.

Corollary: *All observables are integers.*

4 Duality

The formulation of URMT, in the coordinates x,y,z, can be equally formulated in terms of their dual variable forms, $\alpha,\beta,\gamma \in \mathbb{Z},\ (\alpha,\beta,\gamma) \neq (0,0,0)$ and their conjugates $\overline{\alpha},\overline{\beta},\overline{\gamma}$.

Notes

The postulates represent where a fully unified URMT is headed. They are tentative and drafted with both the current and future development of the theory in mind.

Under 'Pythagoras conditions' (discussed later), URMT adheres to all of these postulates.

When not under Pythagoras conditions, the current 3x3 matrix formulation of general URMT, without any conditions, also adheres to most aspects of all four postulates, but the coordinate conjugates $\overline{x}, \overline{y}, \overline{z}$ and their dual forms are not currently used.

1. The Dynamical Conservation Equation

It is not currently specified what the three families of variables P, Q, R, x, y, z, and α, β, γ represent but, being a physical theory, all equations must be dimensionally consistent. Suffice to note, for one possible interpretation, the families can be consistently attributed to acceleration, velocity and position, or force and momentum, hence the usage of the term energy (per unit mass), as follows.

Defining the Kinetic term K and Potential term V as

$$K = P\overline{P} + Q\overline{Q} + R\overline{R}\ ,\ V = \frac{(PQR + \overline{P}\,\overline{Q}\,\overline{R})}{C},$$

then the DCE is written in Kinetic and Potential form as

$$C^2 = K + V\ ,$$

where the constant C is an eigenvalue and invariably set to unity,

$$C = 1\ .$$

This is because the theory can be developed using a unity eigenvalue, and simply extended for a non-unity eigenvalue. Its presence in equations is retained to enable checks on dimensional consistency, i.e. homogeneity of equations (mostly of quadratic degree).

2.1 The Invariance Principle

This is effectively a form of translation or rotation invariance, i.e. you can translate or rotate the dynamical variables whilst preserving the DCE and its solution in the coordinates x, y, z. Note that the 'coordinates' are not necessarily physically associated with position, they can, for now, be any physical quantity.

2.2 Invariance Transformations

The coordinates x, y, z are common to both P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ excepting that P, Q, R are raised by a factor of the coordinates and $\overline{P}, \overline{Q}, \overline{R}$ lowered.

The transformations are given in terms of the standard coordinates x, y, z, not their conjugates $\overline{x}, \overline{y}, \overline{z}$. They could be related to the conjugates instead, but never mixed, even though the dynamical variables and their conjugates are mixed. This is related to the form of the eigenvector solutions, which arise later when applying the transformations to the DCE.

The $\eta, \delta, \varepsilon$ 'local' variations are generally unique, non-zero integers, and can be applied separately, or in combination.

As a general note, in URM3, conjugate and dual quantities are not used much and URMT remains with the twelve variables P, Q, R, $\overline{P}, \overline{Q}, \overline{R}$, x, y, z and α, β, γ. However, all three families and their conjugates are appropriate in the more unified approach to URMT.

3. Observables

This postulate makes URMT Quantum Mechanics friendly and is related to the isomorphism between the dynamical variables, which are integer unity roots in number theory, and the complex roots of unity.

The postulate is the URMT equivalent of saying that all observables are 'real', not complex quantities, and all equations comprise terms such as $\left(P + \overline{P}\right)$ or $P\overline{P}$, just like complex quantities whereby $2\,\mathrm{Re}(Z) = Z + Z^*$ or $\left|Z\right| = +\sqrt{ZZ^*}$.

Conjugation is its own inverse, e.g. $\overline{\overline{P}} = P$, and similarly for the conjugate of a product, e.g. $\overline{P}\overline{\overline{P}} = \overline{P}P = P\overline{P}$ since the dynamical variables are real (actually integers), which naturally commute.

More complicated examples includes the expression for the Potential, $V = (PQR + \overline{P}\overline{Q}\overline{R})$, which is 'real' since $\overline{P}\overline{Q}\overline{R}$ is the conjugate of PQR and hence the sum is real.

For vectors, this postulate is a restatement that all observables and invariants are sums, or sums of products, of vectors and their conjugates, and all invariants are a tensorial contraction.

The corollary does not rule out real or complex numbers as intermediaries in URMT. Their necessity is an open question. However, complex integers, the rational quantity $1/2$ and the irrational quantity $\sqrt{2}$ may be desirable in extensions to the current theory. This is currently open. In these unpublished extensions, the complex integers appear as complex, zero norm vectors, i.e. complex Pythagorean n-tuples such as

triples, quadruplets etc. The rational quantity $1/2$ and irrational quantity $\sqrt{2}$ appear in the scaling of eigenvectors.

4. Duality

URMT, as formulated, gives equations that are symmetric upon interchange of the standard variables with their dual variable equivalents ($\alpha = \tilde{x}, x = \tilde{\alpha}$, $\beta = \tilde{y}, y = \tilde{\beta}$, $\gamma = \tilde{z}, z = \tilde{\gamma}$ and likewise for their conjugates), but give asymmetric solutions when actually solving these equations, i.e. the solution in the coordinates is unique and distinct from the simultaneous solution in the dual variables. This ensures there are three distinct solutions for the coordinates, dynamical variables and dual variables, each with their own unique conjugates. In essence, URMT is formulated from the invariance principle for which a dual equivalent applies; albeit both principles and their respective invariance transformations cannot be applied simultaneously. If they could, the standard and dual solutions would be identical and of insufficient complexity, i.e. of little use.

Note that the dynamical variables and their conjugates are self-dual, i.e. $P = \tilde{\tilde{P}}, \tilde{P} = \tilde{\tilde{P}}$ etc. and duality is really a dual relationship between coordinates x, y, z and dual variables α, β, γ (also known as divisibility or scale factors).

General

The restriction to integers, right from the start, is not actually necessary and can be done at a later stage. However, since the end results are in integers, the postulates have been written assuming integers.

2 The Dynamical Equations

By applying the local variations to the DCE, three dynamical equations and their solutions are obtained when equating the variational terms to zero.

The three linear variational terms, give the dynamical equations.

$$Cx = Ry + \overline{Q}z$$
$$Cy = \overline{R}x + Pz$$
$$Cz = Qx + \overline{P}y$$

Defining the Unity Root Matrix symbol $\mathbf{A}$ as

$$\mathbf{A} = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix},$$

and coordinate Vector $\mathbf{X}_+$ (also simply referred to as $\mathbf{X}$ in [1])

$$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

then the dynamical equations are written in matrix form as

$$\mathbf{A}\mathbf{X}_+ = C\mathbf{X}_+ .$$

By equating the quadratic variational terms to zero, three of nine possible forms of solutions are obtained as follows, only two of which are truly independent,

$$z(C\overline{R} + PQ) = y(CQ + \overline{R}\overline{P})$$
$$z(CR + \overline{P}\overline{Q}) = x(C\overline{P} + QR)$$
$$y(C\overline{Q} + RP) = x(CP + \overline{Q}\overline{R}) .$$

An eigenvector solution is

$$\mathbf{X}_+ = \left(\frac{x}{(C^2 - P\overline{P})} \right) \begin{pmatrix} C^2 - P\overline{P} \\ C\overline{R} + PQ \\ CQ + \overline{R}\overline{P} \end{pmatrix}.$$

Any awkward, zero denominators, e.g. when $C^2 = P\overline{P}$, above, can be removed by applying an invariance transformation without changing the coordinate vector solution $\mathbf{X}_+$.

3 Integers, Co-primality and Dual Variables

The dynamical equations, have nine possible forms of solution, only two of which are independent. Nevertheless, by asserting the following Co-primality Criteria (three criterion $\gcd(x,y)=1,\ \gcd(x,z)=1,\ \gcd(y,z)=1$, in one)

$$\gcd(x,y,z)=1,$$

then, amongst the solutions, this implies that, for some integers α, β and γ, termed **divisibility** (or **scale**) **factors**, the following relations exist

$$(C^2 - P\overline{P}) = \alpha x$$
$$(C^2 - Q\overline{Q}) = \beta y$$
$$(C^2 - R\overline{R}) = \gamma z$$

Notice the defining equations are symmetric upon interchange of the dual variables α with x, β with y and γ with z.

In fact, α, β and γ are variables **dual** to the coordinates x, y, z and vice-versa.

$$\tilde{x} = \alpha,\ \tilde{y} = \beta,\ \tilde{z} = -\gamma,$$
$$x = \tilde{\alpha},\ y = \tilde{\beta},\ z = -\tilde{\gamma},$$

They form the co-vector (eigenrow-vector) $\mathbf{X}^+$

$$\mathbf{X}^+ = \begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}, \alpha,\beta,\gamma \in \mathbb{Z},\ (\alpha,\beta,\gamma) \neq (0,0,0)$$

$\mathbf{X}^+$ is the dual-conjugate of $\mathbf{X}_+$ (or $\mathbf{X}$) and a basis vector in the reciprocal space of vector $\mathbf{X}_+$, satisfying the eigenrow-vector equation

$$\mathbf{X}^+\mathbf{A} = C\mathbf{X}^+$$

However, $\mathbf{X}^+$ is not orthogonal to $\mathbf{X}_+$.

Note that $\mathbf{X}$ and $\mathbf{X}_+$ get used interchangeably in general URMT. However, $\mathbf{X}_+$ is always used when talking about URMT under Pythagoras conditions, see further.

4 The Potential Equation

Summing the defining relations for the divisibility factors α, β and γ, and using the DCE to substitute for the Kinetic term K in terms of the Potential term V, gives

$$+2C^2 = \alpha x + \beta y + \gamma z - (PQR + \overline{P}\,\overline{Q}\,\overline{R})/C$$

The inner product (contraction) of the coordinate vector $\mathbf{X}_+$, and its dual-conjugate $\mathbf{X}^+$, is one of the terms (or contraction of $\mathbf{X}_+$ and conjugate $\overline{\mathbf{X}}_-$, in accord with Postulate 4)

$$\mathbf{X}^+ \cdot \mathbf{X}_+ = \alpha x + \beta y + \gamma z = \overline{\mathbf{X}}_- \cdot \mathbf{X}_+$$

and, using the definition of the Potential $V = (PQR + \overline{P}\,\overline{Q}\,\overline{R})/C$, this gives the Potential equation in vector form as

$$\mathbf{X}^+ \cdot \mathbf{X}_+ = 2C^2 + V$$

Reminder: The eigenvalue C is invariably set to unity, $C = 1$, since URMT, for a non-unity eigenvalue, can be obtained from the unity solution. Its presence is retained for dimensional consistency.

This Potential term and its vanishing, or otherwise, is of fundamental importance in URMT. Note too that the determinant of the unity root matrix $\mathbf{A}$ is the product of the Potential V and the eigenvalue

$$\det(\mathbf{A}) = VC = PQR + \overline{P}\,\overline{Q}\,\overline{R}$$

5 Congruence Relations and Unity Roots

Having made the transition to the integer domain, the dynamical equations are restated as six, linear congruences,

$$Cy \equiv Pz \,(\mathrm{mod}\, x)$$
$$Cz \equiv \overline{P}y \,(\mathrm{mod}\, x)$$

$$Cz \equiv Qx \,(\mathrm{mod}\, y)$$
$$Cx \equiv \overline{Q}z \,(\mathrm{mod}\, y)$$

$$Cx \equiv Ry \,(\mathrm{mod}\, z)$$
$$Cy \equiv \overline{R}x \,(\mathrm{mod}\, z)$$

This is not a unique restatement of the dynamical equations, but the quotients are found to be simply the dual variables (divisibility/scale factors) α, β, γ.

These congruences are exponentiated to integer, order $n >= 2$, and the dynamical variables assigned the following unity root properties ('unity' when $C = \pm 1$, or 'power residues' $\pm C^n$ when $C \neq \pm 1$)

$$P^n \equiv +C^n \,(\mathrm{mod}\, x)$$
$$Q^n \equiv +C^n \,(\mathrm{mod}\, y)$$
$$R^n \equiv -C^n \,(\mathrm{mod}\, z)$$

$$\overline{P}^n \equiv +C^n \,(\mathrm{mod}\, x)$$
$$\overline{Q}^n \equiv +C^n \,(\mathrm{mod}\, y)$$
$$\overline{R}^n \equiv -C^n \,(\mathrm{mod}\, z).$$

With these conditions, the congruences become

$$y^n \equiv z^n \,(\mathrm{mod}\, x)$$
$$z^n \equiv x^n \,(\mathrm{mod}\, y)$$
$$x^n \equiv -y^n \,(\mathrm{mod}\, z).$$

There is no general analytic solution giving unity roots for arbitrary exponent. However, unity roots can be obtained algorithmically and it is, in principal, possible to determine them for any exponent and modulus.

6 The Coordinate Equation

For integer exponent $n >= 2$, some polynomial $k(x, y, z)$, subject to co-primality criteria $\gcd(x, y, z) = 1$, integer coordinates x, y, z, the following coordinate equation satisfies the nth order, congruence relations

$$0 = x^n + y^n - z^n + xyz.k(x, y, z) .$$

The coordinate equation is a Diophantine equation with solutions that can be obtained algorithmically, and is thus amenable to study. A trivial solution, valid for all odd exponents, is $z = x + y$. A cubic solution is $(x, y, z) = (9,31,70)$, $n = 3$, see further.

The coordinate equation is currently a diversionary aside in URMT, and current research is focussed on the quadratic case $n = 2$ and, in particular, when polynomial $k(x, y, z) = 0$ and the coordinate equation reduces to that of Pythagoras.

For $n > 2$, $k \neq 0$, (A. Wiles 1995), the polynomial factor k remains analytically unspecified. However, a priori knowledge is not particularly required as it is derived backward from knowing x, y, z and, indeed, k is not explored further in this paper. In the aforementioned cubic example, $(x, y, z) = (9,31,70)$, then $k = 16$.

7 Conjugate Relations

By the properties of unity roots, the conjugates variables are related to their standard counterparts by the following congruences, termed 'conjugate relations'

$$\overline{P} \equiv C\left(\frac{P}{C}\right)^{n-1} (\text{mod}\, x) \text{ or } \overline{P} \equiv \frac{P^{n-1}}{C^{n-2}} (\text{mod}\, x)$$
$$\overline{Q} \equiv C\left(\frac{Q}{C}\right)^{n-1} (\text{mod}\, y) \text{ or } \overline{Q} \equiv \frac{Q^{n-1}}{C^{n-2}} (\text{mod}\, y)$$
$$\overline{R} \equiv -C\left(\frac{R}{C}\right)^{n-1} (\text{mod}\, z) \text{ or } \overline{R} \equiv \frac{R^{n-1}}{C^{n-2}} (\text{mod}\, z)$$

If $n = 2$ then $\overline{P} \equiv P (\text{mod}\, x)$ etc, which is a very special case in URMT since it reduces the DCE to a Hyperbolic equation and, with the congruence removed (e.g. $\overline{P} = P$ etc.), to Pythagorean triples as two of the three eigenvectors.

For a unity eigenvalue, $C = 1$, $\overline{P} \equiv P^{n-1} (\text{mod}\, x)$, which is the commonly used form.

For a unity eigenvalue, the conjugation of a dynamical variable is therefore one of raising it to the power $n-1$ and obtaining its residue, modulo a coordinate, excepting a specific choice of sign.

For a non-unity eigenvalue, although not obvious from the above relations, there are no divisibility issues in dividing the dynamical variables by the eigenvalue C, using the methods given in [1], paper #6..

8 Conservation Equations & Invariants

Thus, so far, under general URMT there are three key equations (the third when using the unity root extensions to exponent n) and, consequently, three Invariants 0, C^2 and $2C^2$ or 0, +1 and +2 when $C=1$.

The Dynamical Conservation Equation (DCE)

$$+C^2 = P\overline{P} + Q\overline{Q} + R\overline{R} + (PQR + \overline{P}\,\overline{Q}\,\overline{R})/C\,,$$

The Potential equation

$$+2C^2 = \alpha x + \beta y + \gamma z - (PQR + \overline{P}\,\overline{Q}\,\overline{R})/C\,.$$

The coordinate equation

$$0 = x^n + y^n - z^n + xyz.k(x,y,z)\,.$$

Under Pythagoras conditions, there are a further three unique conservation equations.

9 Cubic Example, Unity Eigenvalue

Eigenvector solution

$$n=3,\ x=9,\ y=31,\ z=70,\ C=1,\ \mathbf{X}=\begin{pmatrix}9\\31\\70\end{pmatrix}.$$

Dynamical variables P,Q,R and their conjugates $\overline{P},\overline{Q},\overline{R}$

$$P=-2,\ Q=-6,\ R=-11$$
$$\overline{P}=+4,\ \overline{Q}=+5,\ \overline{R}=+19.$$

The kinetic term $K=P\overline{P}+Q\overline{Q}+R\overline{R}=-247$

The Potential $V=PQR+\overline{P}\,\overline{Q}\,\overline{R}=248$

The Dynamical Conservation Equation

$$+1=K+V=-247+248$$

The dynamical equations $\mathbf{AX}=\mathbf{X}$ in matrix form, unity eigenvalue $C=1$

$$\begin{pmatrix}9\\31\\70\end{pmatrix}=\begin{pmatrix}0&-11&+5\\+19&0&-2\\-6&+4&0\end{pmatrix}\begin{pmatrix}9\\31\\70\end{pmatrix}$$

The Potential equation

$$+2=\alpha x+\beta y+\gamma z-(PQR+\overline{P}\,\overline{Q}\,\overline{R}),$$

$$+2=(+1)(+9)+(+1)(+31)+(+3)(+70)-248.$$

The coordinate equation

$$0=x^n+y^n-z^n+xyz.k(x,y,z)$$

$$0=9^3+31^3-70^3+9.31.70.k(x,y,z),\ k(x,y,z)=+16$$

The divisibility factors α, β and γ

$$(1-P\overline{P})=\alpha x,\ \alpha=+1$$
$$(1-Q\overline{Q})=\beta y,\ \beta=+1$$
$$(1-R\overline{R})=\gamma z,\ \gamma=+3$$

The co-vector $\mathbf{X}^+ = (\alpha \quad \beta \quad \gamma)$

$$\mathbf{X}^+ = (1 \quad 1 \quad 3)$$

The dual dynamical equations $\mathbf{X}^+\mathbf{A} = \mathbf{X}^+$ in matrix form, unity eigenvalue $C = 1$

$$(1 \quad 1 \quad 3)\begin{pmatrix} 0 & -11 & +5 \\ +19 & 0 & -2 \\ -6 & +4 & 0 \end{pmatrix} = (1 \quad 1 \quad 3).$$

The Potential Equation in Vector Form

$$\mathbf{X}^+ \cdot \mathbf{X}_+ = 2 + V$$

$$(1 \quad 1 \quad 3)\begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix} = 250 = 2 + 248$$

Unity Root Properties

$$P^n \equiv +1 \,(\mathrm{mod}\, x),\ -2^3 \equiv +1 \,(\mathrm{mod}\, 9)$$
$$Q^n \equiv +1 (\mathrm{mod}\, y),\ -6^3 \equiv +1 \,(\mathrm{mod}\, 31)$$
$$R^n \equiv -1 (\mathrm{mod}\, z),\ -11^3 \equiv -1 \,(\mathrm{mod}\, 70)$$
$$\overline{P}^n \equiv +1 (\mathrm{mod}\, x),\ +4^3 \equiv +1 \,(\mathrm{mod}\, 9)$$
$$\overline{Q}^n \equiv +1 (\mathrm{mod}\, y),\ 5^3 \equiv +1 \,(\mathrm{mod}\, 31).$$
$$\overline{R}^n \equiv -1 (\mathrm{mod}\, z),\ +19^3 \equiv -1 \,(\mathrm{mod}\, 70).$$

Conjugate Relations

$$\overline{P} \equiv P^{n-1} \,(\mathrm{mod}\, x),\ +4 \equiv (-2)^2 \,(\mathrm{mod}\, 9).$$
$$\overline{Q} \equiv Q^{n-1} \,(\mathrm{mod}\, y),\ +5 \equiv (-6)^2 \,(\mathrm{mod}\, 31).$$
$$\overline{R} \equiv -R^{n-1} \,(\mathrm{mod}\, z),\ +19 \equiv -(-11)^2 \,(\mathrm{mod}\, 70).$$

10 Global Variations

A special type of invariance transformation, $\eta = \delta$, $\varepsilon = -\delta$, is termed a **Global Pythagoras Variation**, denoted by matrix symbol $\mathbf{\Delta^p}$, where $\mathbf{A} \to \mathbf{A} + \mathbf{\Delta^p}$

$$\mathbf{\Delta}^P = \delta \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ -y & +x & 0 \end{pmatrix},$$

and the dynamical variables P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ transform as follows,

$$\begin{aligned} P &\to P + \delta x, \ \overline{P} \to \overline{P} + \delta x \\ Q &\to Q - \delta y, \ \overline{Q} \to \overline{Q} - \delta y \\ R &\to R + \delta z, \ \overline{R} \to \overline{R} - \delta z. \end{aligned}$$

All invariance transformations, including $\mathbf{\Delta^p}$, leave $\mathbf{X}$ unchanged by definition

$$\mathbf{\Delta}^P \mathbf{X} = 0$$

Substituting into the dynamical conservation equation gives the following linear and quadratic variational terms

δ term

$$\begin{aligned} 0 = {} & x(QR + \overline{Q}\overline{R}) - y(RP + \overline{R}\overline{P}) + z(PQ - \overline{P}\overline{Q}) + \\ & x(\overline{P} + P) - y(\overline{Q} + Q) + z(\overline{R} - R) \end{aligned}$$

δ^2 term

$$0 = x^2 + y^2 - z^2 - xy(R + \overline{R}) + xz(Q - \overline{Q}) + yz(\overline{P} - P).$$

11 The Pythagoras Conditions

Looking at the δ^2, quadratic variational term, under a global Pythagoras variation $\mathbf{\Delta^p}$,

$$0 = x^2 + y^2 - z^2 - xy(R + \overline{R}) + xz(Q - \overline{Q}) + yz(\overline{P} - P),$$

the Pythagoras equation naturally emerges, but nowhere has a quadratic exponent been asserted and it is valid for all exponents $n >= 2$

Thus, under the following the **Pythagoras Conditions**

$$\overline{P} = P,\ \overline{Q} = Q,\ \overline{R} = -R,$$

the Pythagoras equation is obtained $0 = x^2 + y^2 - z^2$.

Note that there is no mention of a quadratic exponent in these conditions.

The unity root matrix simplifies to

$$\mathbf{A} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix}.$$

The Potential $V = (PQR + \overline{P}\overline{Q}\overline{R})$ vanishes, $V = 0$

The DCE reduces to a single kinetic term $K = P\overline{P} + Q\overline{Q} + R\overline{R}$, which is the Hyperbolic equation

$$+C^2 = P^2 + Q^2 - R^2.$$

The Conjugate Relations become identities - i.e. the above Pythagoras Conditions

The Global Pythagoras Variation $\mathbf{\Delta^p}$ preserves the zero Potential and Pythagoras Conditions.

The eigenvalues become the symmetric set

$$\lambda = +C,\ \lambda = 0,\ \lambda = -C.$$

The dual variables α, β and γ also satisfy the Pythagoras equation

$$\alpha^2 + \beta^2 - \gamma^2 = 0.$$

The linear variational δ term reduces to another, new conservation equation

$$xP - yQ - zR = 0.$$

12 The Analytic Pythagoras Solution

URMT, under Pythagoras conditions, is a completely solved problem with an analytic solution for all variables, parameterised by three, arbitrary integers k, l and m.

$$x = 2kl$$
$$y = (l^2 - k^2)$$
$$z = (l^2 + k^2)$$

The divisibility factors and dynamical variables are obtained by solving a linear Diophantine equation in unknown integers s and t, given k and l.

$$+C = ks - lt$$

Two particular, integer solutions s' and t', general solutions s and t, parameterised by m

$$s = s' + ml$$
$$t = t' + mk$$

$$P = -(ks + lt)$$
$$Q = (ls - kt)$$
$$R = -(ls + kt)$$

$$\alpha = -2st$$
$$\beta = (t^2 - s^2)$$
$$\gamma = (t^2 + s^2)$$

Eigenvlaue $C = 1$

l	k	x	y	z	s	t	P	-Q	R	α	β	-γ
2	1	4	3	5	1	0	-1	-2	-2	0	-1	-1
3	2	12	5	13	2	1	-7	-4	-8	-4	-3	-5
4	1	8	15	17	1	0	-1	-4	-4	0	-1	-1
4	3	24	7	25	3	2	-17	-6	-18	-12	-5	-13
5	2	20	21	29	3	1	-11	-13	-17	-6	-8	-10
5	4	40	9	41	4	3	-31	-8	-32	-24	-7	-25
6	1	12	35	37	1	0	-1	-6	-6	0	-1	-1
6	5	60	11	61	5	4	-49	-10	-50	-40	-9	-41
7	2	28	45	53	4	1	-15	-26	-30	-8	-15	-17
7	4	56	33	65	2	1	-15	-10	-18	-4	-3	-5
7	6	84	13	85	6	5	-71	-12	-72	-60	-11	-61
8	1	16	63	65	1	0	-1	-8	-8	0	-1	-1
8	3	48	55	73	3	1	-17	-21	-27	-6	-8	-10
8	5	80	39	89	5	3	-49	-25	-55	-30	-16	-34
8	7	112	15	113	7	6	-97	-14	-98	-84	-13	-85
9	2	36	77	85	5	1	-19	-43	-47	-10	-24	-26
9	4	72	65	97	7	3	-55	-51	-75	-42	-40	-58
9	8	144	17	145	8	7	-127	-16	-128	-112	-15	-113

13 Pythagorean Eigenvectors

The standard Eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$, $\mathbf{X}_-$ are

$$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \mathbf{X}_0 = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix}, \mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}.$$

Conjugate eigenvectors

$$\mathbf{X}^+ = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$$
$$\mathbf{X}^0 = \begin{pmatrix} P & -Q & -R \end{pmatrix}$$
$$\mathbf{X}^- = \begin{pmatrix} x & y & -z \end{pmatrix}.$$

Eigenvector Equations

$$\mathbf{A}\mathbf{X}_+ = C\mathbf{X}_+,\ \lambda = +C,\ \mathbf{X}^+\mathbf{A} = C\mathbf{X}^+$$
$$\mathbf{A}\mathbf{X}_0 = 0,\ \lambda = 0,\ \mathbf{X}^0\mathbf{A} = 0$$
$$\mathbf{A}\mathbf{X}_- = -C\mathbf{X}_-,\ \lambda = -C,\ \mathbf{X}^-\mathbf{A} = -C\mathbf{X}^-$$

Eigenvector dot product, invariants (**the six conservation equations**)

$$\mathbf{X}_+ \cdot \mathbf{X}^- = x^2 + y^2 - z^2 = 0 \text{ Pythagoras}$$
$$\mathbf{X}_- \cdot \mathbf{X}^+ = \alpha^2 + \beta^2 - \gamma^2 = 0 \text{ Pythagoras}$$
$$\mathbf{X}_0 \cdot \mathbf{X}^0 = P^2 + Q^2 - R^2 = +C^2 \text{ the DCE}$$
$$\mathbf{X}_+ \cdot \mathbf{X}^+ = \mathbf{X}_- \cdot \mathbf{X}^- = \alpha x + \beta y + \gamma z = +2C^2 \text{ the Potential Equation } (V = 0)$$
$$\mathbf{X}_+ \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^- = xP - yQ - zR = 0 \text{ the Delta equation}$$
$$\mathbf{X}_- \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^+ = \alpha P - \beta Q + \gamma R = 0 \text{ the Dual Delta equation}$$

Eigenvector cross products

$$\mathbf{X}^+ \wedge \mathbf{X}^0 = -\mathbf{X}^0 \wedge \mathbf{X}^+ = C\mathbf{X}_-$$
$$\mathbf{X}_+ \wedge \mathbf{X}_0 = -\mathbf{X}_0 \wedge \mathbf{X}_+ = C\mathbf{X}^-$$
$$\mathbf{X}^- \wedge \mathbf{X}^+ = -\mathbf{X}^+ \wedge \mathbf{X}^- = +2C\mathbf{X}_0$$
$$\mathbf{X}_- \wedge \mathbf{X}_+ = -\mathbf{X}_+ \wedge \mathbf{X}_- = +2C\mathbf{X}^0$$
$$\mathbf{X}^0 \wedge \mathbf{X}^- = -\mathbf{X}^- \wedge \mathbf{X}^0 = C\mathbf{X}_+$$
$$\mathbf{X}_0 \wedge \mathbf{X}_- = -\mathbf{X}_- \wedge \mathbf{X}_0 = C\mathbf{X}^+.$$

The eigenvector triple product is also an invariant with a value of $+2C^3$, hence all three eigenvectors are linearly independent and form a basis.

$$\mathbf{X}^+ \wedge \mathbf{X}^0 \cdot \mathbf{X}^- = \mathbf{X}_- \cdot \mathbf{X}^- = \mathbf{X}_+ \wedge \mathbf{X}_0 \cdot \mathbf{X}_- = \mathbf{X}^- \cdot \mathbf{X}_- = +2C^3$$

14 The T Operator & Minkowski '2+1'

By defining the transformation matrix '$\mathbf{T}$ Operator' as

$$\mathbf{T} = \mathbf{T}^T = \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then, under Pythagoras conditions, the conjugate vectors are formed from their standard counterparts $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ as follows

$$\begin{aligned} \mathbf{X}^- &= (\mathbf{T}\mathbf{X}_+)^T \\ \mathbf{X}^0 &= (\mathbf{T}\mathbf{X}_0)^T . \\ \mathbf{X}^+ &= (\mathbf{T}\mathbf{X}_-)^T \end{aligned}$$

Of course, the $\mathbf{T}$ operator is simply the Minkowski metric tensor (in '2+1' Special Relativity), raising or lowering the indices of a tensor. In this case, converting the components of a vector in a standard space, to its components in the reciprocal space, i.e.

$$\begin{aligned} \mathbf{T} &\cong \eta_{ij} = \eta^{ij} , \\ \mathbf{X}^{-i} &= \eta^{ij}\mathbf{X}_{+j} \text{ etc} \end{aligned}$$

Note that the transpose operation, in the above, is part of the raising/lowering process when using matrix, vector notation, as opposed to purely tensorial notation.

It is of note that the Minkowski metric is that of flat space; the concept of curvature in the URMT eigenvector space is introduced later when studying the eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ as a highly oblique basis

15 Plotting the Eigenvectors

The eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ are equivalent to ordered triples of the coordinates x, y, z, dynamical variables P, Q, R, and divisibility factors α, β, γ as in

$$\mathbf{X}_+ \cong (x, y, z),\ \mathbf{X}_0 \cong (P, -Q, R),\ \mathbf{X}_- \cong (\alpha, \beta, -\gamma),$$

All three triples can be plotted as points p $p \in \mathbb{Z}^3$, on the familiar three-axis, right-handed, Cartesian set x, y, z with α, P plotted on the x axis, β, Q plotted on the y axis, and γ, R plotted on the z axis.

Plotting in this way, each eigenvector is a vector with its base at the origin and tip either on a cone or hyperboloid - these are discussed shortly.

This is not the only way to plot the eigenvectors. As mentioned, three parameters k l and m characterise all three eigenvectors and we could, instead, draw three Cartesian axes k l and m at which every point (k, l, m), is a basis set of the three eigenvectors giving a 3D vector field. This concept is more useful when thinking of eigenvectors as an evolving, oblique triad, see later.
.

16 Two Cones

Since $\mathbf{X}_+$ and $\mathbf{X}_-$ are Pythagorean triples their geometry is that of a discrete cone - a separate cone for each eigenvector. Each cone also divides into two halves, upper and lower, defined as follows:

The **upper cone**, symbol $\mathbf{C}_U$, comprises the set of all Pythagorean triples for which $z, \gamma > 0$.

The **lower cone**, symbol $\mathbf{C}_L$, comprises the set of all Pythagorean triples for which $z, \gamma < 0$.

The **cone**, symbol $\mathbf{C}$, is the union of sets $\mathbf{C}_L$ and $\mathbf{C}_U$, $\mathbf{C} = \mathbf{C}_L \cup \mathbf{C}_U$.

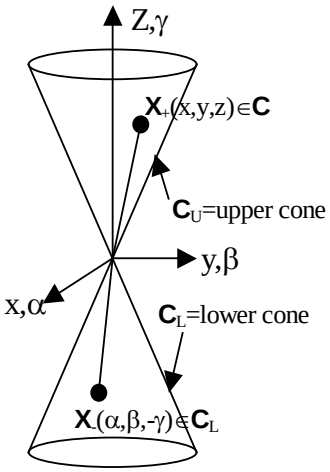


Figure 1 Upper and Lower Cones

Given $\mathbf{X}_+$ and $\mathbf{X}_-$ are defined as follows,

$$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ \mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}.$$

then $\mathbf{X}_+ \in \mathbf{C}_U$ for $z > 0$ and therefore $\mathbf{X}_- \in \mathbf{C}_L$ for $\gamma > 0$, both conditions being met when using the analytic solutions $z = (l^2 + k^2)$ and $\gamma = (t^2 + s^2)$, i.e. if $\mathbf{X}_+$ is in $\mathbf{C}_U$ then $\mathbf{X}_-$ is simultaneously in $\mathbf{C}_L$.

17 A Hyperboloid

Since the DCE is a hyperboloid, the geometry of $\mathbf{X}_0$ is that of a discrete hyperboloid sheet, divided into an upper and lower half as follows.

The **Upper hyperboloid**, symbol $\mathbf{H}_L$, is the set of all points $p = (P,-Q,R)$ where $R > 0$.

The **lower hyperboloid**, symbol $\mathbf{H}_L$, is the set of all points $p = (P,-Q,R)$ where $R \le 0$.

The **hyperboloid**, symbol $\mathbf{H}$, is the union of sets $\mathbf{H}_L$ and $\mathbf{H}_U$

$$\mathbf{H} = \mathbf{H}_L \cup \mathbf{H}_U .$$

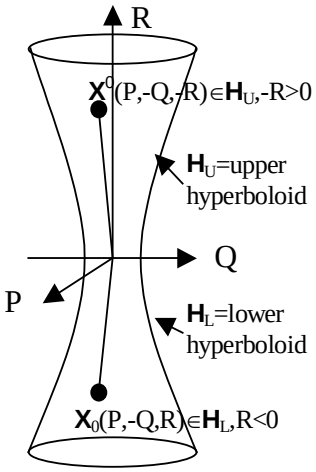


Figure 2 The Hyperboloid

18 The Lattice

Since all points p on the cone and hyperboloid are ordered triples of integers, i.e. $p \in \mathbb{Z}^3$, the cone and hyperboloid quadric 'surface' is not a continuum but an infinite set of points, coincident with a continuous, quadric 2D surface in $\mathbb{R}^3$. The set of points is collectively referred to as the lattice $\mathbf{L}$.

The **lattice,** is defined as the set of points $\mathbf{L}$ formed from the union of all points in the cone $\mathbf{C}$ and the hyperboloid $\mathbf{H}$

$$\mathbf{L} = \mathbf{C} \cup \mathbf{H}.$$

In other words, the lattice is a collective term for all discrete points occupied by $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$, their conjugates, $\mathbf{X}^-$, $\mathbf{X}^0$ and $\mathbf{X}^+$ respectively.

19 No Singularities or Infinities

Exclusion of the origin from the Lattice **L**.

The origin is excluded from the lattice, i.e. sets **C**, **H** for genuine algebraic reasons rather than just an arbitrary condition, and is explained as follows.

As regards the cone **C**, it excludes the zero Pythagorean triple $(0,0,0)$ because, if z is zero in $\mathbf{X}_+$, then the divisibility relation $(C^2 - R\overline{R}) = \gamma z$ becomes $(C^2 + R^2) = 0$ ($\overline{R} = -R$ under Pythagoras conditions) and, since $C^2 > 0$, this cannot hold true. Note that $C \neq 0$ because it is a non-zero eigenvalue by definition - the theory already has a separate, zero eigenvalue.

As regards **H**, it also excludes the origin as algebraically impossible since the eigenvector elements $\mathbf{X}_0$ (and $\mathbf{X}^0$) satisfy the hyperbolic DCE, for which $(P,Q,R) = (0,0,0)$ is not a valid solution. Basically, **H** has a non-zero radius at the origin.

Geometrically speaking, exclusion of the origin from **L** means that the cones are without a tip and the hyperboloid always has a non-zero radius in the '$x-y$' plane.

Hence, **C** and **H** are referred to as having zero singularity.

As a consequence, any path of $\mathbf{X}_-$ or $\mathbf{X}_+$ connecting the cones would skip the origin when going from $\mathbf{C}_L$ to $\mathbf{C}_U$ (evolving forward) and vice versa, $\mathbf{C}_U$ to $\mathbf{C}_L$ (evolving backward); likewise for a path on the hyperboloid **H**.

No Infinities or zero divisors.

Zero divisors are possible within URMT, e.g. in the $\mathbf{X}_+$ eigenvector denominator when $C^2 = P\overline{P}$. But, since such denominators contain dynamical variables, the zero can be removed by transformation, without altering $\mathbf{X}_+$. Albeit, $\mathbf{X}_0$ and $\mathbf{X}_-$ are transformed harmlessly. This transformation property is very useful for any awkward expressions, in particular the indeterminate form $0/0$, which can often arise.

20 Eigenvector Evolution

Using the analytic solution, under Pythagoras conditions, then the eigenvector $\mathbf{X}_+$ and point (x, y, z) is completely characterised by the arbitrary integers k, l.

If k and l are fixed, then $\mathbf{X}_+$ is considered static and represents a single point in the upper cone $\mathbf{C}_U$ for $z > 0$ (convention).

The $\mathbf{X}_0$ and $\mathbf{X}_-$ triples, $(P, -Q, R)$ and $(\alpha, \beta, -\gamma)$ respectively, are obtained by solving a linear Diophantine equation adding a third, arbitrary integer parameter m.

IMPORTANT The parameter m is the same as $-\delta$ in the global Pythagoras transformation $\mathbf{\Delta}^{\mathbf{p}}$

Thus, for a static $\mathbf{X}_+$, fixed integer parameters k and l, the eigenvectors $\mathbf{X}_0$ and $\mathbf{X}_-$ evolve with respect to $\mathbf{X}_+$ according to parameter m.

Denoting the initial (primitive) $m = 0$ eigenvector solution by a prime, then the general eigenvector solutions, in terms of m, can be written as

$$\mathbf{X}_{m+} = \mathbf{X}'_+ = \mathbf{X}_+ \text{ (static - no } m \text{ dependence)}$$
$$\mathbf{X}_{m0} = -m\mathbf{X}_+ + \mathbf{X}'_0$$
$$\mathbf{X}_{m-} = -m^2\mathbf{X}_+ + 2m\mathbf{X}'_0 + \mathbf{X}'_- .$$

Since $\mathbf{X}_{m0}$ and $\mathbf{X}_{m-}$ are parameterised by m, the eigenvectors trace out an evolving path (trajectory) through the lattice, on the hyperboloid $\mathbf{H}$ (for $\mathbf{X}_{m0}$) and cone $\mathbf{C}$ (for $\mathbf{X}_{m-}$) for each static point $\mathbf{X}_+$ in the cone, i.e. each point on the $\mathbf{X}_+$ cone has another cone associated with it for $\mathbf{X}_{m-}$, and a hyperboloid for $\mathbf{X}_{m0}$.

Both $\mathbf{X}_0$ and $\mathbf{X}_-$ become anti-parallel to $\mathbf{X}_+$ in the large m limit, with $\mathbf{X}_-$ lying on the mirror image cone to $\mathbf{X}_+$, and $\mathbf{X}_0$ becoming ever closer, but never touching, the same mirror image cone in $\mathbf{C}$.

$$\mathbf{X}_{m0} \approx -m\mathbf{X}_+ , \ \mathbf{X}_{m-} \approx -m^2\mathbf{X}_+ , \ |m| >> 0$$

Since the sets $\mathbf{C}$ and $\mathbf{H}$ are disjoint, i.e. $\mathbf{C} \cap \mathbf{H} = \varnothing$, the paths of $\mathbf{X}_-$ ($\mathbf{X}_- \in \mathbf{C}$) and $\mathbf{X}_0$ ($\mathbf{X}_0 \in \mathbf{H}$) never intersect, i.e. they never contain a common point.

21 Free-fall and Null Geodesics

Looking at the $\mathbf{X}_{m-}$ evolution equation

$$\mathbf{X}_{m-} = -m^2\mathbf{X}_{+} + 2m\mathbf{X}'_0 + \mathbf{X}'_{-},$$

then, for large m, it is seen that each increment $\delta\mathbf{X}_{m-}$, in $\mathbf{X}_{m-}$ approximates a simple, scalar multiple $-2m$ of $\mathbf{X}_{+}$

$$\delta\mathbf{X}_{m-} \approx -2m\mathbf{X}_{+},\ |m| >> 0.$$

Since each $\mathbf{X}_{+}$ is a Pythgorean triple with zero norm, the evolving path of $\mathbf{X}_{m-}$ ever closer approximates a null geodesic as m grows larger.

If the m parameter is associated with time, then we can consistently use 'Force, Potential' terminology and state that the position vector $\mathbf{X}_{m-}$ traces a trajectory through the lattice, with an inverse square law curvature path (with respect to time, i.e. $\propto 1/m^2$),

The cone in which $\mathbf{X}_{m-}$ resides is a zero-Potential surface (Pythagoras conditions $V = 0$) of constant energy ($= C^2$ by the DCE), there is no Kinetic/Potential energy interchange and, thus, no force acting upon it. Therefore $\mathbf{X}_{m-}$ effectively moves in free-fall.

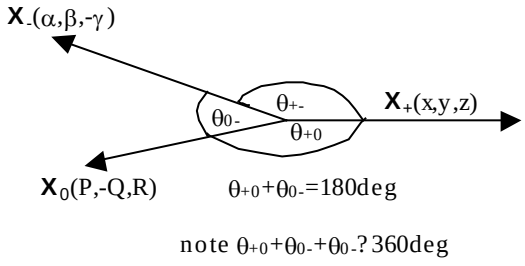
22 Angular Evolution and Flatness

Although the eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ ($\cong \mathbf{X}_{m0}$) and $\mathbf{X}_-$ ($\cong \mathbf{X}_{m-}$) are linearly independent, they are far from orthonormal, i.e. they are oblique and each of non-unit length.

Defining the **flatness parameter** ω as the ratio of the eigenvalue C to the dynamical variable R, for $R \neq 0$

$$\omega = C/R,\ R \neq 0.$$

and denoting the angle between $\mathbf{X}_+$ and $\mathbf{X}_0$ by θ_{+0}, the angle between $\mathbf{X}_+$ and $\mathbf{X}_-$ by θ_{+-}, and the angle between $\mathbf{X}_0$ and $\mathbf{X}_-$ by θ_{0-},



then θ_{+0}, θ_{+-} and θ_{0-} are obtained by the standard inner product relations, e.g. $\cos\theta_{+0} = \mathbf{X}_+ \cdot \mathbf{X}_0 / |\mathbf{X}_+||\mathbf{X}_0|$, where $Sg(a) = sign(a)$

$$\cos\theta_{+0} = +Sg(zR)\left(2/(2+\omega^2)\right)^{1/2}$$
$$\cos\theta_{+-} = -1/(1+\omega^2)$$
$$\cos\theta_{0-} = -Sg(\gamma R)\left(2/(2+\omega^2)\right)^{1/2}.$$

The flatness is inversely proportional to dynamical variable R, for fixed eigenvalue C (energy C^2).

Since R is parameterised by the evolution parameter m, as in $R = R' - mz$, then, $\omega = C/(R' - mz)$ and, for large m, and to first order in $1/m$, this approximates to

$$\omega \approx \left(\frac{-C}{z}\right)\frac{1}{m},\ |m| >> 1.$$

Whatever finite value for C and z (static) is chosen, a value for m, and consequently R, can always be found such that $|R| >> C$ and $|\omega| << 1$ by (8.13), hence,

$$\lim_{m\to\pm\infty} \omega = 0.$$

Thus, the flatness parameter ω becomes ever smaller as the evolution progresses, i.e. as the evolutionary parameter m grows ever larger.

The smaller ω, the flatter the eigenvector basis, with perfect flatness given by $\omega = 0$,

The choice of the name 'flatness' is evident in that $\cos\theta_{+-}$ converges to –1 and $\cos\theta_{+0}$ also converges to -1 (for $z > 0$, $R < 0$), and so θ_{+-} and θ_{+0} converge to 180 deg as $|\omega|$ becomes ever smaller, i.e. the axes flatten out.

The flatness is inversely proportional to the evolutionary parameter m. A larger m means a flatter basis, i.e. the longer m evolves, the flatter the basis and lattice*.

* Note, strictly speaking, it is the eigenvector basis that flattens, but since this property is virtually independent of any static starting vector $\mathbf{X}_+$ (for sufficiently large m), it applies to every trajectory originating from every $\mathbf{X}_+$, and so it is also the lattice which is referred to as flattening with increasing m.

The flatness parameter ω is proportional to the finite eigenvalue C (or ω^2 proportional to energy C^2). Hence, the larger the eigenvalue (energy), the longer it takes to achieve same flatness.

Example Angle Data

$\mathbf{X}_+ =$ (4,3,5)	$\mathbf{X}_{m0}$ ($C = 1$)			$\mathbf{X}_{m-}$				$\theta_{+0} + \theta_{0-} = 180$	
m	P	$-Q$	R	α	β	$-\gamma$	θ_{+-}	θ_{+0}	θ_{0-}
0	-1	-2	-2	0	-1	-1	143.130	160.529	19.471
1	-5	-5	-7	-6	-8	-10	168.522	174.232	5.768
2	-9	-8	-12	-20	-21	-29	173.267	176.628	3.372
8	-33	-26	-42	-272	-225	-353	178.071	179.035	0.965
32	-129	-98	-162	-4160	-3201	-5249	179.500	179.750	0.250

Example Flatness Data

evolution parameter	flatness parameter ω	ω approx-imated from $\bar{\theta}_{+-}$	% error
m	$\omega = C/R$, ($C = 1$)	$\hat{\omega}_{+-} = \frac{-\bar{\theta}_{+-}}{\sqrt{2}}$	$100\left\|\frac{\hat{\omega}_{+-} - \omega}{\omega}\right\|$
0	-0.5000	-0.4550	9.00
1	-0.1429	-0.1417	0.84
2	-0.0833	-0.0831	0.29
8	-0.023810	-0.023804	0.024
32	-0.00617284	-0.00617274	0.0016

23 Curvature

Curvature, symbol κ, is defined as the rate of change of angle θ_{+-} with respect to the evolution parameter m, i.e.,

$$\kappa = \delta\theta_{+-} / \delta m .$$

Changes in angle, $\delta\theta_{+-}$, on each evolutionary tick, δm, are given by the discrete difference $\delta\theta_{+-} = \theta_{(m)+-} - \theta_{(m-1)+-}$, with $\delta m = 1$, the curvature is thus given by

$$\kappa = \frac{\delta\theta_{+-}}{\delta m} = \theta_{(m)+-} - \theta_{(m-1)+-} .$$

Using $\cos\theta_{+-} = -1/(1+\omega^2)$ and $\omega = C/(R' - mz)$, and approximating for large m (small ω), gives the curvature as follows

$$\kappa \approx \left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m^2}, \quad |m| >> 1.$$

The curvature for θ_{+-}, with respect to m, is an inverse square law.

The same relation, barring a factor, applies to angles θ_{+0} and θ_{0-} respectively.

Therefore all three angles θ_{+-}, θ_{0-} and θ_{+0} have an inverse square law curvature relation with respect to m, for large m.

Evolutionary parameter m can be time, length (e.g. arc length) or other?

The expression for κ is a 'large m' approximation, becoming better as $|m|$ increases.

The curvature is proportional to the eigenvalue C, which is effectively a free parameter for tuning.

Example Curvature Data

evolution parameter m	θ_{+-} (deg)	angle θ_{+-} (rad)	curvature κ (9.2) $\theta_{(m)+-} - \theta_{(m-1)+-}$	$\hat{\kappa}$ approximated, for $C=1$, $z=5$ $\hat{\kappa} \approx \left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m^2}$	% error $100\left\|\frac{\hat{\kappa}-\kappa}{\kappa}\right\|$
0	143.130	2.498092	-	-	-
1	168.522	2.941258	0.443166	0.2828427	36.0
2	173.267	3.024081	0.0828232	0.0707107	15.0
8	178.071	3.107929	0.00454656	0.00441942	2.8
32	179.500	3.132863	0.00027800	0.00027621	0.64

24 Physical Scale

If URMT is a theory of Physics, then it operates at the Planck level, and its smallest unit, i.e. one, is considered to be the equivalent Planck unit, be it length, time or, perhaps, mass.

A scale can be obtained by looking at the smallest possible interval of evolution, i.e. $m=1$. At the smallest, Planck scale of physical reality, a value of $m=1$ represents a length of 1.6×10^{-35} m or a time of 5.4×10^{-44} s. So, at the one metre length, m is $\approx10^{35}$ and, for a time of 1s, m is $\approx10^{43}$. Thus, there is no issue with approximating the continuous by the discrete with these sizes of numbers.

The smallest 'almost trivial' solution for a Pythagorean triple, $\mathbf{X}_+$ or $\mathbf{X}_-$ in URMT, is $(0,1,1)$, which has a magnitude of $\sqrt{2}$ and is thus considered the Planck level, i.e. if it represented distance it would be around 10^{-35} m.

When making physical comparisons on the flatness ω and curvature κ of the vector space, the question arises, how large does 'large m' have to be? The short answer is not very large, but it is dependent on the value chosen for the conserved quantity given by eigenvalue C.

$$\kappa\approx\left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m^2},\quad \omega\approx\left(\frac{-C}{z}\right)\frac{1}{m},\ |m|>>1.$$

As seen in the above equations, the flatness ω is proportional to C and inversely proportional to m, for constant z. The curvature is also proportional to C and inversely proportional to the square of m.

Although C may be large (but finite) the z component in eigenvector $\mathbf{X}_+$ can be made as large as desired by suitable choice of k and/or l in the analytic solution for $\mathbf{X}_+$, to nullify the effect of a large C. Given $\mathbf{X}_+$ is static then, once chosen, the evolution proceeds as per a small C.

Considering m as units of Planck time, then a value $m=10$ represents $\approx10^{-42}s$, and so the flatness at $10^{-42}s$ is 10% of its initial value at $10^{-43}s$, with the curvature a mere 1% of its initial value.

Whatever the scaling of m, it is clear that a large m does not have to be very large before the flatness and curvature reduce to zero, far less than 1 second.

Translating this discussion on scale to a cosmological analogy then, given that m need not be very large at all, i.e. $m = O(10^1)$, on an evolutionary scale of $m = O(10^{43})$, the flattening is all but over in the very early, inflationary stages (first 10 ticks). Certainly, by 10^{43} ticks (1s), flatness reigns supreme. With a flattening period (a function of the clock ticks m) tamed by choice of a large starting energy C^2, then it is clear that the larger the starting energy (for a fixed z, see above), the longer the evolutionary period to attain flatness.

Associating eigenvalue C with speed, and C^2 as energy per unit mass (kinetic energy/mass), then the Planck unit of C is simply the speed of light, little c, i.e. $C = c \approx 3\times10^8 m/s$. Given the age of the universe is approximately 13 billion years, which equates to about 10^{17} seconds, then the evolution parameter is, in units of Planck time, $m \approx 10^{62}$. Using the definition of the flatness parameter $\omega \approx \left(\frac{-C}{z}\right)\frac{1}{m}$, for large m, then with $z = 1ms^{-2}$ as in the 'almost trivial' solution, and with $C \approx 3\times10^8 m/s$, the flatness is around $\omega \approx 10^{-54}$ (dimensionless), i.e. flat to within 1 part in 10^{54}.

25 Non-zero, Zero-point Energy.

The quantity C^2 in the DCE can be associated with energy (strictly speaking, energy per unit mass), and is split into a Kinetic term K and Potential term V as as a consequence.

$$K = P\overline{P} + Q\overline{Q} + R\overline{R}$$

$$V = \frac{(PQR + \overline{P}\,\overline{Q}\,\overline{R})}{C}.$$

$$C^2 = K + V.$$

Since the DCE equates to the square of a non-zero eigenvalue, it is never zero and its smallest value, $C^2 = +1$, is akin to a 'zero-point' energy.

The zero-point energy for a single oscillator, is given by $E_0 = \hbar\omega/2$ for oscillator frequency ω.

Using the Planck frequency (the reciprocal of the Planck time), this gives the energy value for $C^2 = +1$ as $E \approx 10^{19} GeV$ ($2 \times 10^9 J$).

26 Integer Invariants

The three eigenvalues $\lambda = \pm C, 0$ are, by definition, invariants of the theory under Pythagoras conditions.

The eigenvector space generates six other scalar invariants via the vector, inner product relations between the three eigenvectors and their conjugate forms. Of course, three of these are zero by the orthogonal properties between eigenvector columns and rows.

The full suite of inner products is a set of six equations

$$\mathbf{X}_+ \cdot \mathbf{X}^- = x^2 + y^2 - z^2 = 0 \text{, Pythagoras equation}$$
$$\mathbf{X}_- \cdot \mathbf{X}^+ = \alpha^2 + \beta^2 - \gamma^2 = 0 \text{, Pythagoras equation}$$
$$\mathbf{X}_0 \cdot \mathbf{X}^0 = P^2 + Q^2 - R^2 = +C^2 \text{, Dynamical conservation equation}$$
$$\mathbf{X}_+ \cdot \mathbf{X}^+ = \mathbf{X}_- \cdot \mathbf{X}^- = \alpha x + \beta y + \gamma z = +2C^2 \text{, Potential equation}$$
$$\mathbf{X}_+ \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^- = xP - yQ - zR = 0 \text{, Delta equation}$$
$$\mathbf{X}_- \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^+ = \alpha P - \beta Q + \gamma R = 0 \text{, Dual delta equation}$$

The volume element gives another, derived invariant $+2C^3$

$$\mathbf{X}_+ \wedge \mathbf{X}_0 \cdot \mathbf{X}_- = +2C^3.$$

The important point about these values is that, for any evolved set of eigenvectors $\{\mathbf{X}_+, \mathbf{X}_{m0}, \mathbf{X}_{m-}\}$ and their conjugates, they are truly invariant in the lattice $\mathbf{L}$. They are the same value along every trajectory of $\mathbf{X}_{m0}$ and $\mathbf{X}_{m-}$, for each and every associated $\mathbf{X}_+$.

The invariants cover the integer set $\{-C, 0, C, C^2, 2C^2, 2C^3\}$ and it is noticed that, for unity C, this set covers the most basic integers $\{-1,0,1,2\}$. Even when $C \neq 1$, their ratios also include the simple set of integers $\{0, \pm\frac{1}{2}, \pm 1, \pm 2, \}$.

Furthermore, given the earlier considerations on Physical scale, the integers values involved in the eigenvectors can easily be $O(10^{40})$ and much higher, but the eigenvectors will always be linked by the smallest of integer invariants $0, \pm 1$ etc.

With such small invariants spanning the entire lattice, independent of all parameters, most importantly the evolutionary parameter m, it is tempting to associate them with well-known, conserved physical properties such as charge, spin et al.

27 Eigenvectors as Derivatives

The evolution equations of the eigenvectors are as follows, where the superscript prime denotes the initial, $m = 0$ value

$$\mathbf{X}_{m+} = \mathbf{X}'_{+} = \mathbf{X}_{+} \text{ (static, i.e. no } m \text{ dependence)}$$
$$\mathbf{X}_{m0} = -m\mathbf{X}_{+} + \mathbf{X}'_{0}$$
$$\mathbf{X}_{m-} = -m^2\mathbf{X}_{+} + 2m\mathbf{X}'_{0} + \mathbf{X}'_{-}.$$

The standard calculus derivative $\frac{d}{dm}$ is used as a good, large m approximation for discrete differences, i.e.

$$\frac{d}{dm} \approx \frac{\delta}{\delta m},\ m >> 0,\ \delta m = 1,$$

Differentiating the evolution equations for $\mathbf{X}_{m0}$ and $\mathbf{X}_{m-}$ with respect to evolutionary parameter m, the eigenvector derivatives are

$$\frac{d\mathbf{X}_{+}}{dm} = 0$$

$$\frac{d\mathbf{X}_{m0}}{dm} = -\mathbf{X}_{+}$$

$$\frac{d\mathbf{X}_{m-}}{dm} = 2\mathbf{X}_{m0},\ \frac{d^2\mathbf{X}_{m-}}{dm^2} = -2\mathbf{X}_{+}$$

It is seen that $\mathbf{X}_{+}$ is constant, with zero derivative

The first derivative of $\mathbf{X}_{m0}$ wrt m is simply the constant vector $-\mathbf{X}_{+}$ and

The first derivative of $\mathbf{X}_{m-}$ wrt m is just twice the vector $\mathbf{X}_{m0}$, and therefore the second derivative of $\mathbf{X}_{m-}$ wrt m is just twice the constant vector $-\mathbf{X}_{+}$.

Thus, the eigenvectors span zero, first and second order derivatives, to within a constant factor ± 2. This constant can be scaled-out as eigenvectors are arbitrary to within a scale factor.

28 Physical Association of Eigenvectors

From the previous, Calculus section, the lattice basis vectors $\mathbf{X}_{+}$, $\mathbf{X}_{m0}$ and $\mathbf{X}_{m-}$ are seen to naturally represent constant, first and second order integrals of $\mathbf{X}_{+}$ with respect to evolutionary parameter m.

So, for example, if parameter m is associated with time, then $\mathbf{X}_{m-}$ is position, $\mathbf{X}_{m0}$ is velocity and $\mathbf{X}_{+}$ constant acceleration.

$$\mathbf{X}_{+},\ x,y,z \cong \text{acceleration}$$
$$\mathbf{X}_{m0},\ P,Q,R,C \cong \text{velocity}$$
$$\mathbf{X}_{m-},\ \alpha,\beta,\gamma \cong \text{position.}$$

Since $\mathbf{X}_{m0}$ (velocity) is an eigenvector comprising dynamical variables P,Q,R, this is entirely consistent with associating dynamical variables as velocity.

Of course, the parameter could also be length, mass or other combination of mass, length and time. In fact, it is suspected the same theory may have a different physical interpretation, dependent upon the units associated with m.

29 Newtonian Physics

Associating the evolving eigenvectors as follows, to an acceleration vector $-\mathbf{a}$, velocity vector $\mathbf{v}$ and position vector $\mathbf{x}$, and treating the parameter m as time t,

$$\mathbf{X}_{+} = -\mathbf{a}/2$$
$$\mathbf{X}_{m0} = \mathbf{v}/2$$
$$\mathbf{X}_{m-} = \mathbf{x}$$
$$m = t,$$

then the evolution equations simply translate to Newtonian Physics

$$\frac{d\mathbf{v}}{dt} = \mathbf{a}$$

$$\mathbf{v} = \mathbf{a}t + \mathbf{v}'$$

$$\mathbf{x} = \frac{\mathbf{a}}{2}t^2 + \mathbf{v}t + \mathbf{x}'$$

Thus, if twice the Pythagorean triple $\mathbf{X}_{+}$ represents a constant acceleration ($-\mathbf{a}$), then twice $\mathbf{X}_{m0}$ represents a velocity $\mathbf{v}$, and $\mathbf{X}_{m-}$ (unscaled) represents the position $\mathbf{x}$, which is also a Pythagorean triple. $\mathbf{X}_{m0}$ itself, satisfies the Hyperbolic DCE.

Of course, scaling $\mathbf{X}_{+}$ by -1 (legal for eigenvectors) can make it represent a 'positive' acceleration $-\mathbf{a}$, so the sign is immaterial.

For example if the acceleration is $-\mathbf{g}$, e.g. for an object thrown upward in the earth's constant gravitational field, then the familiar, high-school equations are obtained.

$$\frac{d\mathbf{v}}{dt} = -\mathbf{g}$$

$$\mathbf{v} = -\mathbf{g}t + \mathbf{v}'$$

$$\mathbf{x} = -\frac{\mathbf{g}}{2}t^2 + \mathbf{v}t + \mathbf{x}'$$

30 Duality

It has been mentioned prior that the divisibility factors (or scale) factors α, β and γ are dual to the coordinates x, y, z and that all equations in URMT, under Pythagoras conditions, are symmetric upon interchange of the dual variables α with x, β with y and γ with z. You can formulate URMT in x, y, z or in α, β, γ, albeit not simultaneously.

Most importantly though, not only do the dual variables α, β, γ form the co-vector $\mathbf{X}^+$ in the reciprocal space, they also form the eigenvector $\mathbf{X}_{m-}$ in the standard basis, alongside $\mathbf{X}_+$ and $\mathbf{X}_{m0}$. So $\mathbf{X}_{m-}$ is the dual eigenvector of $\mathbf{X}_+$, i.e. $\mathbf{X}_{m-} = \tilde{\mathbf{X}}_+$ and vice-versa, $\mathbf{X}_+ = \tilde{\mathbf{X}}_{m-}$.

At the same time, $\mathbf{X}_{m-}$ evolves with respect to $\mathbf{X}_+$ according to parameter m and, for large m

$$\mathbf{X}_{m-} \approx -m^2 \mathbf{X}_+, \ |m| >> 0$$

Thus, the vector $\mathbf{X}_{m-}$ tends to look like $\mathbf{X}_+$, scaled by m^2 and, barring scale, the two worlds $\mathbf{X}_{m-}$ and $\mathbf{X}_+$ look the same.

Thus, by studying the world of $\mathbf{X}_{m-}$, then simply rescaling by m^2, gives the world of $\mathbf{X}_+$ and vice versa.

In terms of the null-cone sets $\mathbf{C}_L$ and $\mathbf{C}_U$ then, since $\mathbf{X}_- \in \mathbf{C}_L$ when $\mathbf{X}_+ \in \mathbf{C}_U$, this represents a duality between the small and large-scale geometry of the sets expressed as $\mathbf{C}_L = \tilde{\mathbf{C}}_U$ and $\mathbf{C}_U = \tilde{\mathbf{C}}_L$.

$m,\frac{1}{m}$ Duality and $\mathbf{X}_{m0}$

The middle ground (macroscopic world) is considered to be that of the eigenvector $\mathbf{X}_{m0}$, residing in the disjoint, hyperbolic set $\mathbf{H}$.

With this interpretation, relative to $\mathbf{X}_{m0}$, the microscopic region is $\mathbf{X}_{+}$ and the large scale region that of $\mathbf{X}_{m-}$ so, for large m,

$$\begin{aligned}\mathbf{X}_{+}&,\ \text{micro}\\ \mathbf{X}_{m0} &\approx -m\mathbf{X}_{+},\ \text{macro}\\ \mathbf{X}_{m-} &\approx -m^2\mathbf{X}_{+},\ \text{large}\end{aligned}$$

Dividing throughout by m

$$\begin{aligned}\frac{1}{m}&\mathbf{X}_{+}\\ \mathbf{X}_{m0} &\approx \mathbf{X}_{+}\\ \mathbf{X}_{m-} &\approx -m\mathbf{X}_{+}\end{aligned}$$

When viewed with respect to $\mathbf{X}_{m0}$, for large m

$\mathbf{X}_{+}$ tends to $\frac{1}{m}\mathbf{X}_{+}$
$\mathbf{X}_{m-}$ tends to $m\mathbf{X}_{+}$,

$\mathbf{X}_{m0}$ sees an $m,\frac{1}{m}$ duality between the microscopic and the very large.

The $m,\frac{1}{m}$ duality is considered analogous to mirror manifold symmetry in modern mathematical physics.

31 Winding Number

The parameter m is analogous with a winding number since it controls the quotient in a moduli relation (congruence).

For example, the dynamical variable P and its initial quantity (or particular solution) P', satisfy the following congruence property

$$P \equiv P' (\mathrm{mod}\, x),$$

by virtue of its definition as invariance transformation, in accord with the invariance principle.

Expanding this congruence gives

$$P = P' - mx.$$

Thus, the evolutionary parameter m is effectively a quotient of the coordinate and equivalent to a winding number since P changes in amounts of mx.

32 Symmetry Breaking

Under Pythagoras conditions, the Potential term in the URM is zero and, consequently, the three eigenvalues are also symmetric, centred on a zero eigenvalue, i.e. $\lambda = \pm C, 0$ with the two eigenvectors, $\mathbf{X}_+$ and $\mathbf{X}_-$, both Pythagorean triples.

However, applying a local invariance transformation, for three, arbitrary integer $\gamma, \delta, \varepsilon$ values, destroys this eigenvalue symmetry and the zero potential, leaving a non-zero Potential, non-symmetric eigenvalues, and only one Pythagorean eigenvector $\mathbf{X}_+$ invariant, by definition. Contrast this with a global Pythagoras variation $\mathbf{\Delta}^{\mathbf{p}}$, which preserves the zero Potential.

For real eigenvalues, under a local transformation, the Potential must be negative and hence, by the DCE, the Kinetic energy increases at the expense of the potential energy decreasing. In other words, the zero Potential state represents the highest energy, most symmetric state of URMT and its solutions, and any change to the potential breaks this symmetry.

With a non-zero Potential, the eigenvector $\mathbf{X}_-$ is no longer a Pythagorean triple and, geometrically speaking, the second cone $\mathbf{C}_L$ (lower by convention), which is effectively a mirror image of the $\mathbf{X}_+$ cone $\mathbf{C}_U$, is replaced by a more general lattice structure, the symmetry in the geometry is therefore also broken.

Given the Potential goes from zero and invariant, to non-zero and non-invariant, under such a local transformation, there is a kinetic/Potential energy interchange, and thus a force induced. **So, URMT introduces an extra force by a symmetry breaking mechanism.**

The zero potential energy state is considered analogous to the ubiquitous physical analogy of a ball sitting on the top of a circularly symmetric hat. The system has perfect circular symmetry but is unstable and, as soon as the ball falls, it picks out a preferred direction and the symmetry is broken. Naturally, since it falls, it acquires more and more kinetic energy as the potential energy decreases. Performing a global Pythagoras transformation, under Pythagoras conditions, is equivalent to rotating the hat, which remains symmetric with the ball sitting on top at zero Potential; apply a local transformation and the Pythagoras conditions are destroyed, the eigenvalue and eigenvector, lattice symmetry is lost, and so too the zero Potential.

33 Symmetric Equations, Asymmetric Solutions

The equations of URMT, under Pythagoras conditions, are symmetric upon interchange of x, y, z with α, β, γ. One formulation is the dual of the other, and working in one will give the other and vice-versa.

However, the solution space is very different, especially at the Planck level, i.e. for a small evolutionally parameter m, the solutions for x, y, z and α, β, γ, as represented by eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_{m-}$, are very different excepting both are distinct, Pythagorean triples.

Nevertheless, as m grows ever larger, the vector $\mathbf{X}_{m-}$ converges to look like $\mathbf{X}_+$, scaled by m^2, i.e.

$$\mathbf{X}_{m-} \approx -m^2 \mathbf{X}_+, \text{ for large } m.$$

For example, the simplest, non-trivial $(4,3,5)$ solution is, for $m = 0$,

$$\mathbf{X}_+ = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, \ \mathbf{X}_- = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

and, for $m = 64$,

$$\mathbf{X}_+ = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, \ \mathbf{X}_- = \begin{pmatrix} -16512 \\ -12545 \\ -20737 \end{pmatrix} \approx -64.4^2 \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, \text{ to within 1\%}$$

It is of note that no matter how large α, β, γ grow with m, the Potential equation, linking the two triples x, y, z and α, β, γ, remains invariant with a value $+2C^2$ since, under Pythagoras conditions, the Potential is zero and invariant ($V = 0$), i.e.

$$\mathbf{X}_+ \cdot \mathbf{X}^+ = \mathbf{X}_- \cdot \mathbf{X}^- = \alpha x + \beta y + \gamma z = +2C^2$$

and, using the above numbers,

$$(-16512)(4) + (-12545)(3) + (20737)(5) = +2 \ \ (C = 1)$$

Note that $\gamma = +20737$, $-\gamma$ appears in $\mathbf{X}_-$, but $+\gamma$ appears in $\mathbf{X}^+$.

34 Unifying Concepts

The unified approach currently pertains to URMT when under Pythagoras conditions, which relates ALL three families of standard variables x, y, z, P, Q, R, α, β, γ, with their conjugates $\overline{x}, \overline{y}, \overline{z}$, $\overline{P}, \overline{Q}, \overline{R}$, $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$

$$\begin{gathered}\overline{x} = x,\ \overline{y} = y,\ \overline{z} = -z,\\ \overline{P} = P,\ \overline{Q} = Q,\ \overline{R} = -R,\ \overline{C} = -C\\ \overline{\alpha} = \alpha,\ \overline{\beta} = \beta,\ \overline{\gamma} = -\gamma,\end{gathered}$$

and also their dual forms $\widetilde{x}, \widetilde{y}, \widetilde{z}$, $\widetilde{P}, \widetilde{Q}, \widetilde{R}$, $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}$,

$$\begin{gathered}\widetilde{x} = \alpha,\ \widetilde{y} = \beta,\ \widetilde{z} = -\gamma,\\ \widetilde{P} = P,\ \widetilde{Q} = Q,\ \widetilde{R} = R,\ \widetilde{C} = -C.\\ \widetilde{\alpha} = x,\ \widetilde{\beta} = y,\ \widetilde{\gamma} = -z,\end{gathered}$$

There are also dual-conjugate forms, not shown here, but they can be deduced from the above.

The unified approach treats both the URM3 $\mathbf{A}$ and the global, Pythagoras variational matrix $\mathbf{\Lambda}^P$, as variational matrices and adds a third, new matrix to give three variational '$\mathbf{A}$' matrices $\mathbf{A}_+$, $\mathbf{A}_0$ and $\mathbf{A}_-$, one for each of the three families of variables.

$$\mathbf{A}_0 = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix},\ \mathbf{A}_+ = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ -y & x & 0 \end{pmatrix},\ \mathbf{A}_- = \begin{pmatrix} 0 & -\gamma & -\beta \\ \gamma & 0 & \alpha \\ -\beta & \alpha & 0 \end{pmatrix},$$

These standard matrix forms also have conjugate and dual forms, related as follows, where conjugation means both transposition and conjugation of their elements

$$\begin{aligned}\overline{\mathbf{A}}_+ &= \mathbf{A}_+, \qquad \overline{\mathbf{A}}_0 = \mathbf{A}_0, \qquad \overline{\mathbf{A}}_- = \mathbf{A}_-,\\ \widetilde{\mathbf{A}}_+ &= \mathbf{A}_-, \qquad \widetilde{\mathbf{A}}_0 = \mathbf{A}_0, \qquad \widetilde{\mathbf{A}}_- = \mathbf{A}_+.\end{aligned}$$

The three standard Pythagorean eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$, also have conjugate and dual forms related as follows

$$\begin{aligned}\overline{\mathbf{X}}_+ &= \mathbf{X}^-, \qquad \overline{\mathbf{X}}_0 = \mathbf{X}^0, \qquad \overline{\mathbf{X}}_- = \mathbf{X}^+,\\ \widetilde{\mathbf{X}}_+ &= \mathbf{X}_-, \qquad \widetilde{\mathbf{X}}_0 = \mathbf{X}_0 \qquad \widetilde{\mathbf{X}}_- = \mathbf{X}_+.\end{aligned}$$

With these conjugate and dual forms, under the unified approach, the conservation equations can be written in a neater, form

For example, the DCE

$$\mathbf{X}_0 \cdot \overline{\mathbf{X}}_0 = P\overline{P} + Q\overline{Q} + R\overline{R} + C\overline{C} = 0$$

The Pythagoras Equation

$$\mathbf{X}_+ \cdot \overline{\mathbf{X}}_+ = x\overline{x} + y\overline{y} + z\overline{z} = 0$$

The Dual Pythagoras Equation

$$\tilde{\mathbf{X}}_+ \cdot \overline{\tilde{\mathbf{X}}}_+ = \tilde{x}\overline{\tilde{x}} + \tilde{y}\overline{\tilde{y}} + \tilde{z}\overline{\tilde{z}} = \alpha\overline{\alpha} + \beta\overline{\beta} + \gamma\overline{\gamma} = 0$$

The Potential Equation

$$\overline{\mathbf{X}}_- \cdot \mathbf{X}_+ + C\overline{C} = 0$$

Given the simplicity of the relationships between standard, conjugate and dual forms, this may seem unnecessary, but it illustrates the symmetry in URMT, when under Pythagoras conditions, and puts it on a familiar footing with notation used in mathematical physics. It is also makes URMT neater and more concise - as for all good notation.

35 Hermitian Operators

All three $\mathbf{A}$ matrices are Hermitian in that they are equal to their transpose conjugate, i.e.

$$\mathbf{A} = \left[\overline{\mathbf{A}}\right]^T = \overline{\mathbf{A}}$$

This is because all conjugate variables are their own inverse, e.g. $\overline{\overline{P}} = P$ etc.

Taking the original URM as an example, then

$$\mathbf{A}^T = \begin{pmatrix} 0 & \overline{R} & Q \\ R & 0 & \overline{P} \\ \overline{Q} & P & 0 \end{pmatrix}, \text{ transposition}$$

$$\overline{\mathbf{A}}^T = \begin{pmatrix} 0 & \overline{\overline{R}} & \overline{Q} \\ \overline{R} & 0 & \overline{\overline{P}} \\ \overline{\overline{Q}} & \overline{P} & 0 \end{pmatrix}, \text{ conjugation of the matrix elements}$$

and using $\overline{\overline{P}} = P$, $\overline{\overline{Q}} = Q$, $\overline{\overline{R}} = R$, then

$$\overline{\mathbf{A}}^T = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix} = \mathbf{A}$$

All eigenvectors and matrices in URMT, under Pythagoras conditions, are Hermitian.

36 Annihilation Equations

The form of the $\mathbf{A}$ matrices is specifically chosen such that they satisfy the following, equations, known as annihilation equations, since they annihilate their related eigenvectors

$$\mathbf{A}_+\mathbf{X}_+ = 0$$
$$\mathbf{A}_0\mathbf{X}_0 = 0$$
$$\mathbf{A}_-\mathbf{X}_- = 0\,.$$

In index notation

$$\mathbf{A}_i\mathbf{X}_i = 0\,,\ i = +,0,-\,.$$

The annihilation property $\mathbf{A}_+\mathbf{X}_+ = 0$ is simply the original variational expression $\boldsymbol{\Delta}^P\mathbf{X}_+ = 0$, restated in the unified notation.

The expression $\mathbf{A}_0\mathbf{X}_0 = 0$ is the existing definition of the eigenvector $\mathbf{X}_0$ for eigenvalue $\lambda = 0$.

The expression $\mathbf{A}_-\mathbf{X}_- = 0$ is new and considered the dual partner to $\mathbf{A}_+\mathbf{X}_+ = 0$, i.e. $\tilde{\mathbf{A}}_+\tilde{\mathbf{X}}_+ = 0$ since $\mathbf{A}_- = \tilde{\mathbf{A}}_+,\ \mathbf{X}_- = \tilde{\mathbf{X}}_+$.

37 Operator Calculus

Comparing the $\mathbf{A}_+$ (formerly $\mathbf{\Delta}^P$) eigenvector transformations (not derived here) with the eigenvector derivative relations

$$\mathbf{A}_+\mathbf{X}_+ = 0\,,\ \frac{d\mathbf{X}_+}{dm} = 0\,,$$

$$\mathbf{A}_+\mathbf{X}_0 = -C\mathbf{X}_+\,,\ \frac{d\mathbf{X}_0}{dm} = -\mathbf{X}_+ \text{ * (see below)},$$

$$\mathbf{A}_+\mathbf{X}_- = 2C\mathbf{X}_0\,,\ \frac{d\mathbf{X}_-}{dm} = 2\mathbf{X}_{m0}\,,$$

it is seen that the $\mathbf{A}_+$ matrix is equivalent to the derivative operator

$$C\frac{d}{dm} = \mathbf{A}_+\,.$$

* Note here that the m subscript has been dropped from the evolving eigenvectors $\mathbf{X}_{m0}$ and $\mathbf{X}_{m-}$ so that they are just written $\mathbf{X}_0$ and $\mathbf{X}_-$, and assumed functions of m.

Likewise, by looking at the $\mathbf{A}_-$ transformations and comparing with the integral form of the derivative relations, it is seen that the $\mathbf{A}_-$ matrix is equivalent to the integral operator

$$\mathbf{A}_- = 2C\int dm\,.$$

Lastly, looking at the $\mathbf{A}_0$ transformations

$$\mathbf{A}_0\mathbf{X}_+ = +C\mathbf{X}_+\,,$$
$$\mathbf{A}_0\mathbf{X}_0 = 0\,,$$
$$\mathbf{A}_0\mathbf{X}_- = -C\mathbf{X}_-\,.$$

these are seen to be just a constant (proportional) multiplier, i.e. the eigenvalue.

Thus, the $\mathbf{A}$ matrices act as derivative, proportional, and integral operators on the eigenvectors

$\mathbf{A}_+$ = derivative operator

$\mathbf{A}_0$ = constant multiplier

$\mathbf{A}_-$ = integral operator

38 Invariance Transformations & Generators

Defining a global skew transformation $\mathbf{\Delta}^s$ as follows

$$\mathbf{\Delta}^s = \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ +y & -x & 0 \end{pmatrix}, \text{ where } \mathbf{\Delta}^s\mathbf{X}_+ = 0 \text{ since } \mathbf{X}_+ \text{ is invariant by definition}$$

then the unity root matrix $\mathbf{A}_0$ transforms as follows, where m is the evolution parameter

$$\mathbf{A}_0 \to \mathbf{A}_0 - m\mathbf{\Delta}^s, \text{ where } (\mathbf{A}_0 - m\mathbf{\Delta}^s)\mathbf{X}_+ = \mathbf{A}_0\mathbf{X}_+ = C\mathbf{X}_+ .$$

Defining three generator matrices $\mathbf{J}_x$, $\mathbf{J}_y$ and $\mathbf{J}_z$ by

$$\mathbf{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & -1 & 0 \end{pmatrix}, \ \mathbf{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix}, \ \mathbf{J}_z = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then the skew variational matrix $\mathbf{\Delta}^s$ can be decomposed in terms of these three generator matrices as

$$\mathbf{\Delta}^s = x\mathbf{J}_x + y\mathbf{J}_y + z\mathbf{J}_z .$$

The set $\mathbf{J}_x$, $\mathbf{J}_y$ and $\mathbf{J}_z$ is closed under commutation as in

$$\left[\mathbf{J}_x\mathbf{J}_y\right] = -\mathbf{J}_z ,$$

$$\left[\mathbf{J}_y\mathbf{J}_z\right] = -\mathbf{J}_x ,$$

$$\left[\mathbf{J}_z\mathbf{J}_x\right] = -\mathbf{J}_y .$$

39 Invariance Transformations as Rotations

The rotation matrices for an active, right-handed rotation of a vector about each of the Cartesian $x\,(\phi)$, $y\,(\psi)$, $z\,(\theta)$ axes are as follows, with their small angle approximations on the right, are

$$\mathbf{R}_x(\phi)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{pmatrix},\ \mathbf{R}_x(\phi)\approx\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \phi\\ 0 & -\phi & 1\end{pmatrix},\ |\phi|<1,$$

$$\mathbf{R}_y(\psi)=\begin{pmatrix}\cos\psi & 0 & -\sin\psi\\ 0 & 1 & 0\\ \sin\psi & 0 & \cos\psi\end{pmatrix},\ \mathbf{R}_y(\psi)\approx\begin{pmatrix}1 & 0 & -\psi\\ 0 & 1 & 0\\ \psi & 0 & 1\end{pmatrix},\ |\psi|<1,$$

$$\mathbf{R}_z(\theta)=\begin{pmatrix}\cos & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix},\ \mathbf{R}_z(\theta)\approx\begin{pmatrix}1 & \theta & 0\\ -\theta & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ |\theta|<1.$$

In this small angle form, the rotations matrices can be re-written in terms of the identity and generator matrices $\mathbf{J}_x$, $\mathbf{J}_y$ and $\mathbf{J}_z$ as follows

$$\mathbf{R}_x(\phi)\approx\mathbf{I}+\phi\mathbf{J}_x,$$

$$\mathbf{R}_y(\psi)\approx\mathbf{I}+\psi\mathbf{J}_y,$$

$$\mathbf{R}_z(\theta)\approx\mathbf{I}+\theta\mathbf{J}_z.$$

Comparing with the generator matrices $\mathbf{J}_x$, $\mathbf{J}_y$ and $\mathbf{J}_z$ it is seen that the global skew transformation $\mathbf{\Delta}^s$ is identical to an infinitesimal rotation with the following correspondence between the coordinates and angles, with a caveat to follow,

$$x\leftrightarrow\phi$$
$$y\leftrightarrow\psi$$
$$z\leftrightarrow\theta$$

The caveat is that the angles are small, hence the x, y, z elements of eigenvector $\mathbf{X}_+$ must also be small. From discussions on 'Physical Scale', indeed, the smallest, non-trivial Pythagorean triple $(4,3,5)$ (or $(3,4,5)$) is, on a Planck scale, tiny.

40 Two Boosts and a Rotation

Whilst the global skew transformation $\mathbf{\Delta}^s$ can be compared with three pure rotations, the global Pythagoras variation $\mathbf{\Delta}^P$ has a slightly different sign structure, which changes the interpretation of the variation, in generator terms.

$$\mathbf{\Delta}^P = \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ -y & +x & 0 \end{pmatrix}, \text{ c.f. } \mathbf{\Delta}^s = \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ +y & -x & 0 \end{pmatrix}$$

The three generator matrices $\mathbf{K}_x$, $\mathbf{K}_y$ and $\mathbf{J}_z$ are defined as follows

$$\mathbf{K}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & +1 & 0 \end{pmatrix}, \mathbf{K}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathbf{J}_z = \begin{pmatrix} 0 & +1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The global Pythagoras variational matrix $\mathbf{\Delta}^P$ is decomposed in terms of these three generator matrices as

$$\mathbf{\Delta}^P = x\mathbf{K}_x + y\mathbf{K}_y + z\mathbf{J}_z$$

The generators $\mathbf{K}_x$ and $\mathbf{K}_y$ are similar to boosts in '2+1' Special Relativity, and the generator $\mathbf{J}_z$ is the rotation matrix generator for a rotation θ about z.

The set $\mathbf{K}_x$, $\mathbf{K}_y$ and $\mathbf{J}_z$ is closed under commutation as in

$$\begin{aligned} \left[\mathbf{K}_x\mathbf{K}_y\right] &= \mathbf{J}_z \\ \left[\mathbf{K}_y\mathbf{J}_z\right] &= -\mathbf{K}_x \\ \left[\mathbf{J}_z\mathbf{K}_x\right] &= -\mathbf{K}_y \end{aligned}$$

Notably, the commutator $\left[\mathbf{K}_x\mathbf{K}_y\right]_z$ of the two generators $\mathbf{K}_x$, $\mathbf{K}_y$ generating a rotation ('precession') $\mathbf{J}_z$.