Geometric and Physical Aspects of Pythagorean Triples as Eigenvectors

Revised, June 2017. First published 2011.

Richard J. Miller.
richard@microscitech.com,
http://www.urmt.org

Abstract

This paper examines the geometric and physical aspects of the vector space formed by three, linearly independent eigenvectors of a special type of integer matrix. The matrix is special because two of its three eigenvectors are distinct Pythagorean triples with a third, integer eigenvector, linking the two and satisfying a hyperbolic equation. The eigenvector space is seen to be a three-dimensional lattice with the geometry of two discrete cones and a hyperboloid. The linear and angular evolution of the eigenvectors in the lattice is examined and the curvature seen to flatten, following an inverse square law as the evolution progresses. A consistent, physical interpretation of the eigenvectors as position, velocity and acceleration is given and links to key concepts in mathematical physics made.

Keywords. Integer Matrices, Pythagorean Triples, Differential Geometry, Relativity.

MSC2010 Mathematical Subject Classification. 14J81, 15A18, 15B36, 53B50, 83A05.

(1) Introduction

This paper is actually a minor revision of that first published 2011 in [1]#3, i.e. Ref. [1], paper number 3). The paper is split into two parts, where the first part, Sections (1) to (10), starts with a review of earlier work, followed by a study of the geometric aspects of unity root matrix theory, when under Pythagoras conditions [2]. The second part, Section (11) onward, considers the physical aspects plus a summary of the entire paper and some example data in Appendices (B) and (C).

(1.0) Definition. A **Pythagorean triple** considered herein comprises any ordered triple (a,b,c), of integers a,b,c that satisfy the Pythagoras equation $0 = a^2 + b^2 - c^2$. This definition is to be interpreted in its loosest sense with the only condition being that $(a,b,c) \neq (0,0,0)$. In other words, a, b and c are allowed to be positive or negative integers, and a may be less than or greater than b; a can be zero, in which case $|b| = |c|$, or b can be zero, in which case $|a| = |c|$. Non-primitive triples are also included, i.e. those such that for non-zero, integer factor k, if (a,b,c) is a Pythagorean triple then so too is (ka,kb,kc). Otherwise, primitive solutions are co-prime, i.e. $\gcd(a,b) = \gcd(a,c) = \gcd(b,c) = 1$.

The subject of Pythagorean triples, as eigenvectors of the following integer matrix (symbol $\mathbf{A}$), was first studied in [2].

(1.1) $$\mathbf{A} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix},\ P,Q,R \in \mathbb{Z},\ (P,Q,R) \neq (0,0,0).$$

Ref. [2] is purely algebraic and gives no insight into what the particular eigenvector space of $\mathbf{A}$ looks like from a geometric viewpoint; neither does it associate any variables or equations with the physical world. The $\mathbf{A}$ matrix is actually a simplification of a more general ‘Unity Root Matrix Theory’ studied in [1], which is primarily derived from concepts in mathematical physics (transformation invariance) but also offers little insight into what the work might actually represent in the physical world. Albeit, some comparisons are made and, given a similarity of the variational matrices to infinitesimal rotation matrices, angular momentum conservation is mooted [2017_1]. This paper makes some geometric and physical observations to remedy this omission, and it will be seen that the eigenvector geometry is that of a discrete, 3D lattice in $\mathbb{Z}^3$, comprising two discrete cones and a hyperboloid through which the eigenvectors trace out an evolving path. The three eigenvectors can, themselves, be associated with a constant vector and a first and second order derivative with respect to an evolutionary parameter (e.g. time) and, hence, may be identified with position, velocity and acceleration vectors or equivalent.

The use of lattices in Physics is, of course, not new. However, much of the work, to the author’s knowledge, appears focussed on the solution of real or complex, differential equations as functions on a discrete lattice. See, for example, Ref. [3]. The work in this paper is, however, exclusively about a particular lattice in $\mathbb{Z}^3$ and not functions on it. The points in the lattice being represented by the eigenvectors as ordered triples.

The work is exclusively in integers, excepting geometric properties such as curvature, which is approximated in the transition from a discrete set of points to a continuum. The subject matter thus overlaps that of physics in integers; see Section (14.1). Because of the quadratic nature of the equations and their solutions, the earlier work in [1] and [2] concerns number theoretic concepts such as quadratic residues, power residues and primitive roots. This applies, in particular, to the hyperbolic, dynamical conservation equation, (2.1.1) further below, and associated dynamical variables P, Q, R, the latter being associated with integer, primitive roots of unity. The reader is therefore referred to any standard number theory text such as [4], which is comprehensive on quadratic issues. Lastly, at its core, are matrices and eigenvectors, and thus the work (more so in [2]) also calls upon some relatively elementary matrix theory for which most under-graduate texts will suffice.

- Part I -

(2) Review

This Section is a short review of the work in [2], itself deriving from [1], and adds an extension for non-unity eigenvalues.

(2.1) Fundamentals.

The matrix elements P, Q, R of $\mathbf{A}$ (1.1) are termed dynamical variables since they satisfy a conservation equation. The form of this conservation equation, when under Pythagoras conditions (see [1]), is given as follows

(2.1.1) $$+1 = P^2 + Q^2 - R^2 .$$

This equation is actually the non-singular matrix condition $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, $\lambda = 0, \pm 1$, and also known as the dynamical conservation equation.

Although matrix $\mathbf{A}$ (1.1) is a simplification of a unity root matrix in [1], it is of interest since it is shown in [2] that, subject to (2.1.1), it has three eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$, with eigenvalues $\lambda = +1$, $\lambda = 0$, $\lambda = -1$ respectively, whereby two of the eigenvectors, $\mathbf{X}_+$ and $\mathbf{X}_-$, are Pythagorean triples. The eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ are defined as follows, in terms of integers x, y, z and α, β, γ,

(2.1.2) $$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; x, y, z \in \mathbb{Z}, \; (x, y, z) \neq (0,0,0)$$

(2.1.3) $$0 = x^2 + y^2 - z^2$$

(2.1.4) $$\mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}, \; \alpha, \beta, \gamma \in \mathbb{Z}, \; (\alpha, \beta, \gamma) \neq (0,0,0)$$

(2.1.5) $$0 = \alpha^2 + \beta^2 - \gamma^2 .$$

The three, integer variables α, β, γ are also referred to as divisibility (or scale) factors.

It is noted that (2.1.3) and (2.1.5) are the equations of a quadric cone when $(x, y, z) \in \mathbb{R}^3$, and (2.1.1) and (2.2.1), below, are the equations of a quadric, hyperboloid sheet when $(P, Q, R) \in \mathbb{R}^3$. This is discussed again shortly.

The third eigenvector $\mathbf{X}_0$ is a function of the elements of matrix $\mathbf{A}$ (1.1)

(2.1.6) $\mathbf{X}_0 = \begin{pmatrix} +P \\ -Q \\ +R \end{pmatrix}$, $P, Q, R \in \mathbb{Z}$, $(P, Q, R) \neq (0,0,0)$.

It is concluded in [2] that every Pythagorean triple, as an eigenvector $\mathbf{X}_+$, can be related to a similar matrix $\mathbf{A}$ (1.1), subject to constraint (2.1.1), with eigenvalues -1, 0 and $+1$.

(2.2) An extension to non-unity eigenvalues.

The eigenvalues need not be restricted to zero and unity, and the results apply equally for arbitrary, integer eigenvalues $\lambda = +C, 0, -C$, $C \in \mathbb{Z}$, $C > 0$, with associated eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$, $\mathbf{X}_-$ respectively. However, all equations in [2] are for a unity eigenvalue and so require adjustment here. The complete, revised set is provided in Appendix (A).

Addition of a variable eigenvalue C has the advantage that it makes all the equations visibly homogeneous and usually quadratic. More importantly, it adds another arbitrary parameter for tuning to the physical world. The only change to the equations given so far is that the constraint (2.1.1) is modified as follows.

(2.2.1) $C^2 = P^2 + Q^2 - R^2$, $C \in \mathbb{Z}$, $C > 0$

It is seen that this is now clearly homogeneous of degree 2. This equation is actually referred to in [1] as the ‘dynamical conservation equation’ and, adding parameter C, modifies the theory from a conserved quantity of unity in (2.1.1) to a conserved quantity of C^2, above. See also Section (14.6).

Notice that, although C is arbitrary, the sum and product of the eigenvalues ($\lambda = 0, \pm C$) is always zero. The zero sum is dictated by the all-zero lead diagonal and consequent trace of $\mathbf{A}$ (1.1) being zero. The product of the eigenvalues is zero because it is equal to the ‘Potential’ term V, see [1], which is zero under Pythagoras conditions [2]. Pythagoras is the zero Potential form of the dynamical conservation equation (2.2.1), which would normally have an extra, non-zero Potential term on the right of (2.2.1). See also Section (14.4).

With eigenvalues $\lambda = +C, 0, -C$, the eigenvectors satisfy the following equations by definition

(2.2.2)

(2.2.2a) $\mathbf{A}\mathbf{X}_+ = +C\mathbf{X}_+$

(2.2.2b) $\mathbf{A}\mathbf{X}_0 = 0$

(2.2.2c) $\mathbf{A}\mathbf{X}_- = -C\mathbf{X}_-$.

When expanded in component form, these equations are referred to in [1] as the 'dynamical equations'. For the $\mathbf{X}_+$ and $\mathbf{X}_-$ eigenvectors, expanding (2.2.2a) and (2.2.2c) gives the following linear equations

(2.2.3)

(2.2.3a) $Cx = Ry + Qz$

(2.2.3b) $Cy = -Rx + Pz$

(2.2.3c) $Cz = Qx + Py$

(2.2.3e) $-C\alpha = R\beta - Q\gamma$

(2.2.3f) $-C\beta = -R\alpha - P\gamma$

(2.2.3g) $C\gamma = Q\alpha + P\beta$.

Multiplying (2.2.3a) by P, (2.2.3b) by Q and (2.2.3c) by R, and summing as follows, gives the useful identity

(2.2.4) $xP - yQ - zR = 0$.

Likewise, multiplying (2.2.3e) by P, (2.2.3f) by Q and (2.2.3g) by R, and summing as follows, gives another useful identity

(2.2.5) $\alpha P - \beta Q + \gamma R = 0$.

The elements P, Q, R of the matrix $\mathbf{A}$ and eigenvector $\mathbf{X}_0$, the three coordinates x, y, z forming eigenvector $\mathbf{X}_+$, and the three elements α, β, γ of the eigenvector $\mathbf{X}_-$, are all related by the following 'divisibility criteria'

(2.2.6)

(2.2.6a) $\alpha x = (C^2 - P^2)$

(2.2.6b) $\beta y = (C^2 - Q^2)$

(2.2.6c) $\gamma z = (C^2 + R^2)$.

Upon summing these three relations, and using (2.2.1), this neatly combines to give the equation and invariant $+2C^2$.

(2.2.7) $\alpha x + \beta y + \gamma z = +2C^2$

This is actually termed the Potential equation in [1], albeit the Potential V is zero for Pythagoras and so the V term is not seen.

(2.2.8) The vector $\mathbf{X}_0$ (2.1.6) is never a Pythagorean triple because, as shown further below in Section (5), the conserved quantity C^2 in (2.2.1) is never zero and, consequently, $P^2 + Q^2 - R^2 \neq 0$. Therefore $\mathbf{X}_0$, comprising P, Q, R, is never a Pythagorean triple. Nevertheless, for a finite value of eigenvalue C, dynamical variables P, Q and R can be made arbitrarily large and come close to being a

Pythagorean triple. This is by virtue of the invariance principle in [1] and consequential equations (2.5.8), further below. The integer m in (2.5.8) can be chosen to be of sufficient magnitude such that $P,Q,R >> C$, and $\mathbf{X}_0$ then approximates as $\mathbf{X}_0 \approx (-mx, my, -mz)^T$, for $m >> 0$, i.e. it approximates the non-primitive triple $m(-x, y, -z)$ for large m. So, in the limit $m \to \infty$, the vector $\mathbf{X}_0$ (and $\mathbf{X}^0$, (2.3.1b) below) limits to a Pythagorean triple for finite C. Because C is considered a conserved, energy-like quantity, it is also regarded to be finite, even if possibly huge. See also Section (14.6).

(2.3) Conjugate Vectors

The following conjugate (or reciprocal) row-vectors $\mathbf{X}^+$, $\mathbf{X}^0$, $\mathbf{X}^-$ are added such that the vector inner product relations, i.e. $\mathbf{X}^i \cdot \mathbf{X}_j \neq 0$, $i = j$ and $\mathbf{X}^i \cdot \mathbf{X}_j = 0$, $i \neq j$, are of a familiar, orthogonal form. See also Section (14.7).

(2.3.1)

(2.3.1a) $\mathbf{X}^+ = (\alpha \quad \beta \quad \gamma)$

(2.3.1b) $\mathbf{X}^0 = (P \quad -Q \quad -R)$

(2.3.1c) $\mathbf{X}^- = (x \quad y \quad -z)$

The conjugate vectors, $\mathbf{X}^+$, $\mathbf{X}^0$ and $\mathbf{X}^-$, are the basis vectors dual to the standard vectors, $\mathbf{X}_-$, $\mathbf{X}_0$ and $\mathbf{X}_+$ respectively, and are also known in the literature as the reciprocal basis.

By defining the conjugation matrix $\mathbf{T}$ as

(2.3.2)
$$\mathbf{T} = \mathbf{T}^T = \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

the conjugate vectors are formed from their standard counterparts $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ as follows

(2.3.3)

(2.3.3a) $\mathbf{X}^- = (\mathbf{T}\mathbf{X}_+)^T$

(2.3.3b) $\mathbf{X}^0 = (\mathbf{T}\mathbf{X}_0)^T$

(2.3.3c) $\mathbf{X}^+ = (\mathbf{T}\mathbf{X}_-)^T$.

Conversely, the standard vectors are obtained from their conjugate forms as follows:

(2.3.4)
(2.3.4a) $\mathbf{X}_+ = (\mathbf{T}\mathbf{X}^-)^T$
(2.3.4b) $\mathbf{X}_0 = (\mathbf{T}\mathbf{X}^0)^T$
(2.3.4c) $\mathbf{X}_- = (\mathbf{T}\mathbf{X}^+)^T$.

Note that the conjugate of $\mathbf{X}_+$ is $\mathbf{X}^-$ and not $\mathbf{X}^+$. Likewise, the conjugate of $\mathbf{X}_-$ is $\mathbf{X}^+$ and not $\mathbf{X}^-$.

Using standard row-eigenvector algebra, the conjugate vectors also satisfy the following, conjugate (transpose) forms of the eigenvector equations (2.2.2)

(2.3.5)
(2.3.5a) $\mathbf{X}^+\mathbf{A} = +C\mathbf{X}^+$
(2.3.5b) $\mathbf{X}^0\mathbf{A} = 0$
(2.3.5c) $\mathbf{X}^-\mathbf{A} = -C\mathbf{X}^-$.

With the complete set of standard and conjugate vectors defined, the standard definitions of norm and magnitude are given next, preceded by a quick definition of the notation used herein to denote inner products between conjugate, row vectors and standard, column vectors.

(2.4.0) **Inner Product Notation**. The inner (or dot) product of a conjugate, row vector with a column vector, giving a scalar result, is usually written in this paper (and general URMT publications) as the product of a conjugate or reciprocal row-vector and a column vector, without the explicit 'dot' notation. For example, the inner product of the row vector $\mathbf{X}^+$ and column vector $\mathbf{X}_-$ is written as $\mathbf{X}^+\mathbf{X}_-$ instead of the more usual $\mathbf{X}^+ \cdot \mathbf{X}_-$, i.e.

$$\mathbf{X}^+\mathbf{X}_- = \mathbf{X}^+ \cdot \mathbf{X}_-.$$

This notation is that of matrix multiplication, whereby a $1 \times n$ element row-vector multiplies a $n \times 1$ element column vector, to give a 1×1 scalar result.

Note the inner products between vectors of the same form, i.e. inner products between column vectors, or inner products between row vectors, e.g. the inner product between $\mathbf{X}_+$ and $\mathbf{X}_-$ still uses the dot notation, i.e. $\mathbf{X}_+ \cdot \mathbf{X}_-$. Without the dot, the vector product would be that of a $1 \times n$ vector with a $1 \times n$ vector to give an $n \times n$ matrix, otherwise known as an outer product. Such products are not required in this paper.

(2.4.1) **Definition**. The **norm** (or **length**) of a vector, using the standard definition of the norm, see [6], is the square root of the inner product of itself with its conjugate, e.g. for $\mathbf{X}_+$ the norm, denoted by $\|\mathbf{X}_+\|$, is given by $\|\mathbf{X}_+\| = \sqrt{\mathbf{X}^+\mathbf{X}_-}$. Normally the positive square root is assumed unless otherwise stated. Since $\mathbf{X}_+$ is a Pythagorean triple, the norm is zero as $\mathbf{X}^{+}\mathbf{X}_{-} = 0$, which is the same as (2.1.3). The same remarks also apply to $\mathbf{X}_{-}$ and its conjugate form $\mathbf{X}^{+}$, but note that the norm of $\mathbf{X}_0$ is non-zero, as given by $\|\mathbf{X}_0\| = \sqrt{\mathbf{X}^0\mathbf{X}_0} = C^2$, which is the same as (2.2.1).

(2.4.2) **Definition**. The **magnitude** of a vector is the positive square root of the inner product of a vector with itself, e.g. $|\mathbf{X}_{+}| = \mathbf{X}_{+}^{T}\mathbf{X}_{+} = \sqrt[+]{x^2 + y^2 + z^2}$, where $\mathbf{X}_{+}^{T}$ is simply the transpose of $\mathbf{X}_{+}$. Hence $|\mathbf{X}_{+}| = \sqrt{2}z$, using (2.1.3). Likewise, $|\mathbf{X}_{-}| = \sqrt{2}\gamma$, using (2.1.5), and $|\mathbf{X}_0| = \sqrt{C^2 + 2R^2}$. Normally the positive root is taken.

For those familiar with bra and ket notation, see [6], the kets are $\mathbf{X}_{+} \cong |\mathbf{X}_{+}\rangle$, $\mathbf{X}_0 \cong |\mathbf{X}_0\rangle$ and $\mathbf{X}_{-} \cong |\mathbf{X}_{-}\rangle$, and the bras are the conjugate forms $\mathbf{X}^{-} \cong \langle\mathbf{X}_{+}|$, $\mathbf{X}^{+} \cong \langle\mathbf{X}_{-}|$ and $\mathbf{X}^{0} \cong \langle\mathbf{X}_0|$. The above inner products are then given by, for example, $\|\mathbf{X}_{+}\| = \sqrt{\langle\mathbf{X}_{+}|\mathbf{X}_{+}\rangle} = \sqrt{\mathbf{X}^{-}\cdot\mathbf{X}_{+}}$.

(2.5) Analytic solution

The starting point for the study of the geometry of the eigenvectors is the analytic solution, in integers, derived in [2] for all unknown variables, i.e. the elements of the eigenvectors. In total there are nine variables separated into three triples (P,Q,R), (x,y,z) and (α,β,γ). A complete solution is obtained when all nine unknowns $\{x,y,z,P,Q,R,\alpha,\beta,\gamma\}$ are determined. note that the invariant eigenvalue C (2.2.1) is a free, integer parameter, and often set to unity.

The Pythagorean triple (x,y,z) is parameterised in the standard textbook form by two arbitrary integers, k and l, subject to the following condition (2.5.1), which allows one, but not both, of x or y to be zero. Neither is there any constraint on $1 \le k < l$ such that $y > x$. However, there is the constraint $\gcd(k,l)\,|\,C$. This is so that the congruence (2.5.3), further below, has integer solutions. The full list of conditions on k and l is thus

(2.5.1) $\quad k,l \in \mathbb{Z},\ (k,l) \ne (0,0),\ \gcd(k,l)\,|\,C.$

The triple (x,y,z) is then given by the familiar Pythagorean parameterisation

(2.5.2)

(2.5.2a) $\quad x = 2kl$

(2.5.2b) $\quad y = (l^2 - k^2)$

(2.5.2c) $\quad z = (l^2 + k^2).$

To solve for (P,Q,R) and (α,β,γ), two more integers s and t are introduced as solutions to the following congruence (a linear Diophantine equation) in integers k and l

(2.5.3) $+C = ks - lt\,,\ s,t \in \mathbb{Z}.$

This congruence is solved by standard methods [4] to obtain two particular, integer solutions s' and t', and general solutions s and t, parameterised by a third, arbitrary parameter m, $m \in \mathbb{Z}$

(2.5.4)
(2.5.4a) $s = s' + ml$
(2.5.4b) $t = t' + mk$.
Note that a super-script prime denotes an initial value in this paper.

Thus, given a particular solution s' and t', there are now three arbitrary parameters k, l and m. Using these parameters, and the general solution in s and t, then (P,Q,R) and (α,β,γ) are obtained from the following relations

(2.5.5)
(2.5.5a) $P = -(ks + lt)$
(2.5.5b) $Q = (ls - kt)$
(2.5.5c) $R = -(ls + kt)$

(2.5.6)
(2.5.6a) $\alpha = -2st$
(2.5.6b) $\beta = (t^2 - s^2)$
(2.5.6c) $\gamma = (t^2 + s^2)$.

Note that one of α or β, but not both, can legitimately be zero, see [2].

(2.5.7)
The solutions (2.5.2) and (2.5.6) do not cover every Pythagorean triple, according to (1.0), without extensions, as noted and discussed in Appendix (D) in [2]. These extensions allow for all non-primitive triples and all sign combinations, and are really added for completeness rather than uniqueness as solutions. However, the study of the geometric aspects in this paper does not require these extensions and the analysis uses only those solutions given by the equations in this Section. Suffice to say, (2.5.2) and (2.5.6) can give some non-primitive triples and sign combinations, just not all of them.

Integer parameter m can be set to zero such that $s = s'$ and $t = t'$ in (2.5.4). The $m = 0$ case is also referred to as the primitive or initial value solution. Denoting the initial, $m = 0$, dynamical variables by P', Q', R', and the divisibility factors by α', β', γ', then the general solution for (2.5.5) and (2.5.6) is expressed in terms of the initial solution and the coordinates x, y, z as follows:

(2.5.8)
(2.5.8a) $P = P' - mx$
(2.5.8b) $Q = Q' + my$

(2.5.8c) $R = R' - mz$

(2.5.9)
(2.5.9a) $\alpha = \alpha' + 2mP' - m^2 x$
(2.5.9b) $\beta = \beta' - 2mQ' - m^2 y$
(2.5.9c) $\gamma = \gamma' - 2mR' + m^2 z$.

Variation of (P, Q, R) and (α, β, γ) in (2.5.8) and (2.5.9), by arbitrary choice of m, constitutes what is known as a 'global Pythagoras variation' in [1]. It has the effect of transforming the $\mathbf{A}$ matrix (1.1) and eigenvectors $\mathbf{X}_0$, $\mathbf{X}_-$ and their conjugates $\mathbf{X}^0$, $\mathbf{X}^+$, but leaves the eigenvector $\mathbf{X}_+$ and its conjugate $\mathbf{X}^-$ invariant, hence $\mathbf{X}_+$ is referred to as the invariant eigenvector [2017_2]. The conservation equation (2.2.1) also remains invariant, as does the determinant of $\mathbf{A}$, which is zero and identical to the Potential V. Physically, the global variation leaves the Potential invariant and zero when $\mathbf{X}_+$ is a Pythagorean triple, see [2] for full details.

This completes the review of papers [1]#1 and [2].

(3) Triples, Points and Eigenvectors

Before discussing the geometry in detail, the following relevant notes are given. In brief, since $\mathbf{X}_+$ and $\mathbf{X}_-$ are Pythagorean triples, (2.1.3) and (2.1.5), their geometry is that of a cone, and the geometry of $\mathbf{X}_0$ is a hyperboloid sheet, (2.2.1).

All points p on the cone and hyperboloid are ordered triples of integers, i.e. $p \in \mathbb{Z}^3$, and so the cone and hyperboloid quadric 'surface' (2.2.1) is not a continuum but an infinite set of points coincident with a continuous, quadric 2D surface in $\mathbb{R}^3$. The set of points is collectively referred to as the lattice L, defined further below, (4.7).

The points and vectors correspond as follows, which is simply a restatement of their definitions without the row and column vector formalism of matrix algebra,

(3.1)

(3.1a) $\mathbf{X}_+ \sim (x, y, z)$, $\mathbf{X}_0 \sim (P, -Q, R)$, $\mathbf{X}_- \sim (\alpha, \beta, -\gamma)$,

(3.1b) $\mathbf{X}^+ \sim (\alpha, \beta, \gamma)$, $\mathbf{X}^0 \sim (P, -Q, -R)$, $\mathbf{X}^- \sim (x, y, -z)$.

All points are plotted, as per normal, on the familiar, right-handed, Euclidean, 3-axis, (x, y, z) frame, oriented such that the z axis is considered pointing upward, normal to the horizontal x, y plane.

The eigenvectors are treated as fixed with their base at the origin, as opposed to free (floating) vectors, and so allows the two terms, points and eigenvectors, to be used interchangeably, i.e. the lattice points are the tips of the eigenvectors. A common eigenvector origin is a somewhat unnecessary constraint but is, nevertheless, for ease of illustration.

The geometric aspects are chiefly, but not exclusively, studied in terms of the standard eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$, and not their conjugates, $\mathbf{X}^-$, $\mathbf{X}^0$ and $\mathbf{X}^+$ respectively. This is because, by definition, one is the dual basis of the other and either is good as the other.

(4) Cones, Hyperboloid and Lattice

Definition. The **upper cone**, symbol $\mathbf{C}_U$, is the set of all discrete points $p=(a,b,c)$, in accordance with the definition of a Pythagorean triple (1.0), and with $c>0$, specified as

(4.1) $$\mathbf{C}_U=\{p \mid p\in\{(a,b,c)\},(a,b,c)\in\mathbb{Z}^3, c>0\}.$$

Definition. The **lower cone**, symbol $\mathbf{C}_L$, is the set of all discrete points $p=(a,b,c)$, in accordance with the definition of a Pythagorean triple (1.0), and with $c<0$, specified as

(4.2) $$\mathbf{C}_L=\{p \mid p\in\{(a,b,c)\},(a,b,c)\in\mathbb{Z}^3, c<0\}.$$

Definition. The **cone**, symbol $\mathbf{C}$, is the union of sets $\mathbf{C}_L$ and $\mathbf{C}_U$

(4.3) $$\mathbf{C}=\mathbf{C}_L\cup\mathbf{C}_U.$$

Definition. The **lower hyperboloid**, symbol $\mathbf{H}_L$, is the set of all discrete points, pointed to by the eigenvector $\mathbf{X}_0$ in the lower plane, $R\le 0$, i.e. all points p, where p is a triple $(P,-Q,R)$, with P, Q and R given by (2.5.5), satisfying (2.2.1), and formally defined as

(4.4) $$\mathbf{H}_L=\{(P,-Q,R)\}, R\le 0.$$

Note that for $R=0$ the lower hyperboloid intersects the *P*, *Q* plane as a circle, radius *C*, by (2.2.1).

Definition. The **upper hyperboloid**, symbol $\mathbf{H}_U$, is defined as per $\mathbf{H}_L$ (4.4), except $R>0$,

(4.5) $$\mathbf{H}_U=\{(P,-Q,R)\},\ R>0.$$

Definition. The **hyperboloid**, symbol $\mathbf{H}$, is the union of sets $\mathbf{H}_L$ and $\mathbf{H}_U$

(4.6) $$\mathbf{H}=\mathbf{H}_L\cup\mathbf{H}_U.$$

Definition. The **lattice,** as referred to herein, is defined as the set of points $\mathbf{L}$ formed from the union of all points in the cone $\mathbf{C}$ (4.3) and the hyperboloid $\mathbf{H}$ (4.6)

(4.7) $$\mathbf{L}=\mathbf{C}\cup\mathbf{H}.$$

The lattice is a collective term for all discrete points occupied by $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ and their conjugates, $\mathbf{X}^-$, $\mathbf{X}^0$ and $\mathbf{X}^+$ respectively.

(4.8) Commentary

Eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ in $\mathbf{C}_U$ represent Pythagorean triples in the upper half plane, and point upward when $z > 0$ and $-\gamma > 0$; eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ in $\mathbf{C}_L$ represent Pythagorean triples in the lower half and point downward when $z < 0$ and $-\gamma < 0$. Note that z and γ are never zero, see Section (5).

By definition, $\mathbf{C}_U$ and $\mathbf{C}_L$ are disjoint subsets of $\mathbf{C}$.

(4.9) $$\mathbf{C}_U \cap \mathbf{C}_L = \varnothing.$$

The cone $\mathbf{C}$ and hyperboloid $\mathbf{H}$ are also disjoint subsets since the points $(P,-Q,R)$ in $\mathbf{H}$ never lie on the cone $\mathbf{C}$, i.e. they are never Pythagorean triples, see (2.2.8). As a consequence, the following set relation is given

(4.10) $$\mathbf{C} \cap \mathbf{H} = \varnothing.$$

$\mathbf{C}$ comprises every Pythagorean triple allowed by definition (1.0) but, as stated in (2.5.7), the equations (2.5.2) and (2.5.6) do not cover every Pythagorean triple. Thus, strictly speaking, only a subset of $\mathbf{C}$ is required in this paper. See also Section (14.10).

Figure 1 illustrates the upper and lower cones, $\mathbf{C}_U$ and $\mathbf{C}_L$ respectively. $\mathbf{X}_+$ is shown in the upper cone $\mathbf{C}_U$, and $\mathbf{X}_-$ simultaneously in the lower cone $\mathbf{C}_L$. As discussed further below, if $\mathbf{X}_+$ is in $\mathbf{C}_U$ then $\mathbf{X}_-$ is simultaneously in $\mathbf{C}_L$ and vice versa.

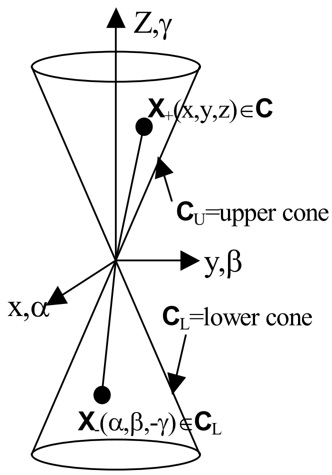

Figure 1 Upper and Lower Cones

Basically, with the usual x, y, z, right-handed coordinate frame oriented such that the z axis is considered pointing upward, normal to the horizontal x, y plane, then $\mathbf{C_U}$ represents an upside-down cone, symmetric about the z axis, with its hypothetical tip at the origin. It is only hypothetical because the origin is excluded from $\mathbf{C_U}$, see Section (5) below, and, as noted, it only comprises discrete, integer points. Conversely, $\mathbf{C}_L$ is the reflection of $\mathbf{C_U}$ in the x, y plane.

The cone tips are shown coincident at the origin but this is not strictly accurate. Indeed, in relativity texts (see [7] for a popular account on light cones), a second cone $\mathbf{C}_L$ is usually drawn with its origin starting at the tip of $\mathbf{X}_+$ on the first cone $\mathbf{C}_U$ and not at the origin of $\mathbf{X}_+$. However, for finite $\mathbf{X}_+$, by a suitably large choice of an integer parameter m in (2.5.9), the $\mathbf{X}_-$ vector can be made such that $|\mathbf{X}_-| >> |\mathbf{X}_+|$ and, in fact, the two cones then become effectively coincident. Therefore, for ease of illustration, the cones $\mathbf{C}_U$ and $\mathbf{C}_L$ are considered with a common origin and mirror images of each other in the x, y plane.

The $\mathbf{X}_+$ and $\mathbf{X}_-$ vectors always lie on opposite cones, e.g. if $\mathbf{X}_+ \in \mathbf{C}_U$ then $\mathbf{X}_- \in \mathbf{C}_L$, i.e. $\mathbf{X}_-$ lies on the mirror image of the cone to $\mathbf{X}_+$ and vice versa. This is by virtue of equation (2.2.6c), $\gamma z = (C^2 + R^2)$, and the eigenvector definitions (2.1.2) and (2.1.4), where the third component, z in $\mathbf{X}_+$, is given the opposite sign to γ in $\mathbf{X}_-$. With the quantity $(C^2 + R^2)$ always greater than zero, then the product γz must also always be positive and, hence, z and γ must be of the same sign. Thus, having a z component in $\mathbf{X}_+$, and a $-\gamma$ component in $\mathbf{X}_-$, always makes the two eigenvectors lie in opposing cones and point in the opposite direction, i.e. away from each other.

For each vector $\mathbf{X}_+$ in a particular quadrant of the $x - y$ plane, the conjugate vector $\mathbf{X}^-$ (2.3.1c) lies in the same quadrant but on the opposite cone. Since the cone slant angle is a constant 45 deg (justification follows), the angle between them is 90 deg and, naturally, their inner vector product is zero, as per (2.3). Identical remarks apply to $\mathbf{X}^+$ and $\mathbf{X}_-$. This also means that if, for example, $\mathbf{X}_+ \in \mathbf{C}_U$, then $\mathbf{X}^- \in \mathbf{C}_L$ and therefore $\mathbf{X}^-$ lies in the same set $\mathbf{C}_L$ as $\mathbf{X}_-$, from what was said earlier. Likewise, $\mathbf{X}^+$ and $\mathbf{X}_+$ also reside in the same cone, $\mathbf{C}_U$ in this example.

The cone slant angle for both cones, $\mathbf{C}_L$ and $\mathbf{C}_U$, is always a constant 45 deg because, considering $\mathbf{X}_+$ for example, at any height z the cone radius r is equal to the height, i.e. $r = \sqrt[+]{x^2 + y^2} = |z|$, hence the cone angle $\theta = \tan^{-1}(z / r) = 45\,\text{deg}$.

Figure 2 illustrates the hyperboloid $\mathbf{H}$ associated with $\mathbf{X}_0$. The same remark for $\mathbf{C}$, about origin positioning, equally applies to $\mathbf{H}$. It too would be rather neater drawn at the tip of $\mathbf{X}_+$ since, just like $\mathbf{X}_-$, for each $\mathbf{X}_+$ there is actually a unique hyperboloid subset $\mathbf{H}_{\mathbf{m-}}$, where $\mathbf{H}_{\mathbf{m-}} \subset \mathbf{H}$, parameterised by the same integer m as for $\mathbf{X}_-$, see $\mathbf{X}_{m0}$ (6.2b). Nevertheless, for ease of illustration, all $\mathbf{H}_{\mathbf{m-}}$ are plotted as if they have a common origin (0,0,0), albeit this origin is not actually an element of $\mathbf{H}$, as explained in the next Section.

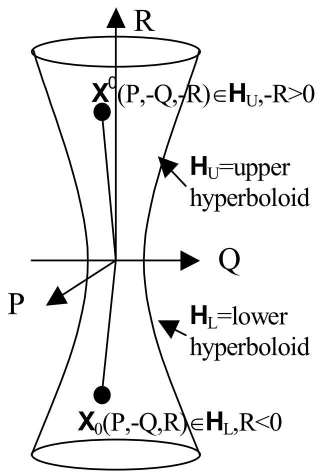

Figure 2 The Hyperboloid

(5) Exclusion of the origin from **L**.

The exclusion of the origin from the sets **C**, **H** and, consequently, **L** (4.7), is for genuine algebraic reasons rather than just an arbitrary condition, and is explained as follows.

As regards **C**, definition (1.0) excludes the zero Pythagorean triple $(0,0,0)$ because it is algebraically impossible within Unity Root Matrix Theory [1]. If z is zero in the Pythagorean triple (x, y, z) then the divisibility relation (2.2.6c) becomes $(C^2 + R^2) = 0$ and, since $C^2 > 0$, this cannot hold true. Note that $C \neq 0$ because it is a non-zero eigenvalue by definition - the theory already has a separate, zero eigenvalue. In general $R \neq 0$ because $R^2 \equiv -1 (\bmod z)$, see [1] or [2]. However it can be zero in what is termed the 'almost trivial' solution, see Appendix (C). Nevertheless, even then, R can be transformed away from zero without affecting (x, y, z) by the invariance principle in [1]. The other triple $(\alpha, \beta, \gamma) \neq (0,0,0)$ because $\gamma \neq 0$, for the same reason that $z \neq 0$, discussed above re (2.2.6c).

As regards **H**, it also excludes the origin as algebraically impossible since the eigenvector elements $\mathbf{X}_0$ (and $\mathbf{X}^0$) satisfy the hyperbolic, conservation equation (2.2.1), for which $(P, Q, R) = (0,0,0)$ is not a valid solution. In general, the dynamical variables P, Q, R are always unity roots but, as mentioned above in the case of **C**, there is a special exception when $Q = R = 0$, $P = \pm C$ or $P = R = 0$, $Q = \pm C$, i.e. triples $(P, Q, R) = (\pm C,0,0)$ or $(P, Q, R) = (0,\pm C,0)$. In this exceptional case, two of the dynamical variables are zero but, nevertheless, the third is never zero, and so the origin remains excluded. Also, as mentioned in [1], if $\mathbf{X}_0$ is unpalatable containing two zeros, these zeros can be transformed away by adjusting m in (2.5.8) and (2.5.9), without affecting the $\mathbf{X}_+$ solution and all invariants of the theory.

Geometrically speaking, exclusion of the origin from **L** means that the cones are without a tip and the hyperboloid always has a non-zero radius in the $x - y$ plane.

See also Section (14.2) and (14.3).

(6) Eigenvector Evolution in L

The general solution for P,Q,R (2.5.8) and α,β,γ (2.5.9) is explicitly parameterised by a single integer m and, hence, eigenvectors $\mathbf{X}_0$ and $\mathbf{X}_-$, by (3.1a), evolve with respect to m. For this reason, m is termed the 'evolution parameter'.

By contrast, the triple (x,y,z) (2.5.2) has no dependency on m and, hence, eigenvector $\mathbf{X}_+$ (3.1a) is static with respect to m. The evolution of $\mathbf{X}_0$ and $\mathbf{X}_-$ will therefore be discussed with respect to $\mathbf{X}_+$. Of course, $\mathbf{X}_+$ is also parameterised by integers k and l (2.5.1) and so too, therefore, $\mathbf{X}_0$ and $\mathbf{X}_-$. A full 3D evolution, given with respect to k, l and m, is beyond the scope of this paper. The benefit in using a single parameter lies in the simplicity of $\mathbf{X}_0$ and $\mathbf{X}_-$ evolving with respect to a single, constant vector $\mathbf{X}_+$, later tentatively identified as a constant acceleration.

Because the evolution of eigenvectors is described by a single parameter, evolving vectors trace a line (path) through L.

Eigenvectors for a specific value of m are denoted by the subscript m, as in $\mathbf{X}_{m+}$, $\mathbf{X}_{m0}$ and $\mathbf{X}_{m-}$. When m is arbitrary, the usual forms $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ are used and the subscript dropped. The subscript m is also dropped for the initial eigenvectors, when $m=0$, and the eigenvectors are given a primed superscript instead, i.e.

(6.1)

(6.1a) $\mathbf{X}'_+ = \mathbf{X}_{m+}(m=0)$

(6.1b) $\mathbf{X}'_0 = \mathbf{X}_{m0}(m=0)$

(6.1c) $\mathbf{X}'_- = \mathbf{X}_{m-}(m=0)$.

Using this notation, the vector form of solutions (2.5.2), (2.5.8) and (2.5.9) becomes

(6.2)

(6.2a) $\mathbf{X}_{m+} = \mathbf{X}'_+$

(6.2b) $\mathbf{X}_{m0} = -m\mathbf{X}_+ + \mathbf{X}'_0$

(6.2c) $\mathbf{X}_{m-} = -m^2\mathbf{X}_+ + 2m\mathbf{X}'_0 + \mathbf{X}'_-$.

The equivalent, conjugate eigenvector forms of (6.2) are obtained by applying the conjugation operator $\mathbf{T}$ (2.3.2) and transposing, as per (2.3.3), to obtain

(6.3)

(6.3a) $\mathbf{X}^{m-} = \mathbf{X}'^{-}$

(6.3b) $\mathbf{X}^{m0} = -m\mathbf{X}'^{-} + \mathbf{X}'^{0}$

(6.3c) $\mathbf{X}^{m+} = -m^2\mathbf{X}'^{-} + 2m\mathbf{X}'^{0} + \mathbf{X}'^{+}$.

The equations (6.2) and (6.3) show how the eigenvectors evolve with respect to m, which could be a spatial, temporal or other evolution parameter. Nevertheless, unsurprisingly, it seems preferable to associate it with a temporal coordinate, i.e. time. A spatial parameter could be associated with the scalar distance, e.g. arc-length, which is also natural for curvature discussions, see Section (9) further below. Nevertheless, because physical association is still rather tentative, time will be mainly considered, and a unit change in m is consequently termed a clock 'tick'. However, an open mind as to its physical interpretation is the safest option. Suffice to add, although m is considered here as increasing in the positive direction, there is no overriding reason for this and it could equally grow more negative. Ultimately, m is just an arbitrary integer parameter to give the most general solution to congruence (2.5.4).

Equation (6.2a) shows that the $\mathbf{X}_+$ eigenvector is independent of m and, for a given choice of two arbitrary integers k and l, fixing the initial eigenvector via (2.5.2) fixes $\mathbf{X}_+$. Thereafter, $\mathbf{X}_+$ does not evolve with respect to m, i.e. it is static, and so

(6.4) $$\mathbf{X}_+ = \mathbf{X}_{m+} = \mathbf{X}'_+ .$$

Since $\mathbf{X}_+ = \mathbf{X}'_+$ the prime is usually dropped from $\mathbf{X}'_+$ and simply denoted by $\mathbf{X}_+$

Equation (6.2b) shows that the $\mathbf{X}_0$ evolution only depends on the eigenvector $\mathbf{X}_+$, which, itself, is static as described above. The $\mathbf{X}_-$ eigenvector's evolution (6.2c) is dependent upon both the initial eigenvectors $\mathbf{X}_+$ and $\mathbf{X}'_0$.

Looking first at $\mathbf{X}_{m0}$ (6.2b), it is clear that for each positive increment (a clock tick) in m, $(m > 0)$, the vector changes by $-\mathbf{X}_+$, i.e. it changes anti-parallel to $\mathbf{X}_+$, see Figure 3. The increment is defined as $\delta\mathbf{X}_{m0}$ and given by the difference of successive iterates $m-1$ and m as follows, using (6.2b),

(6.5) $$\delta\mathbf{X}_{m0} = \mathbf{X}_{m0} - \mathbf{X}_{(m-1)0} = -\mathbf{X}_+ ,\ (m > 0) .$$

Note that if m incremented in the negative direction then the vector change would be parallel to $\mathbf{X}_+$, and not anti-parallel.

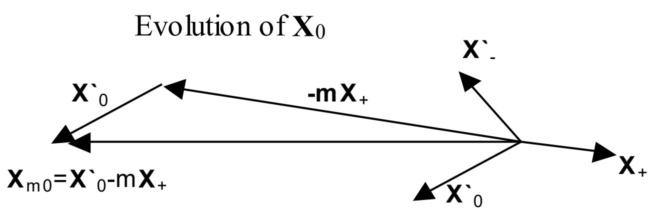

Figure 3 Evolution of X0

As $\mathbf{X}_+$ is static, $\mathbf{X}_0$ lengthens as m increases and aligns with it. For large m it is effectively anti-parallel to $\mathbf{X}_+$, i.e. $\mathbf{X}_{m0} \approx -m\mathbf{X}_+, m >> 0$. Since the initial vector $\mathbf{X}'_0$ is definitely not parallel to $\mathbf{X}_+$ ($\mathbf{X}_0$ is never a Pythagorean triple, including a multiple of it, see (2.2.8)), and neither is it ever in C for the same reason, $\mathbf{X}_0$ becomes more anti-parallel to $\mathbf{X}_+$ as m increases, i.e. its angle with $\mathbf{X}_+$ converges to 180 deg. Whilst $\mathbf{X}_0$ is always in H, it moves ever closer to C because $\mathbf{X}_0$ approximates, ever closer, a Pythagorean triple, see (2.2.8).

Because all changes to $\mathbf{X}_0$ are subtractions of a linear multiple of $\mathbf{X}_+$, the vector $\mathbf{X}_{m0}$ never moves off the plane spanned by the two initial vectors $\mathbf{X}_+$ and $\mathbf{X}'_0$. Therefore the path it traces is always in this plane. This path is thus the intersection of the plane, spanned by $\mathbf{X}_+$ and $\mathbf{X}'_0$, with the hyperboloid H. Furthermore, all points $\mathbf{X}_{m0}$, $m > 0$, are collinear but also always lie in H.

Visually it is hard to see a plane intersecting a curved hyperboloid sheet, where the intersection is also a straight line, as opposed to the more usual conic. However, it is the points in H that are collinear and not the space in-between, which is undefined and not in H. It can be verified that all $\mathbf{X}_{m0}$ points lie in H as follows: writing $\mathbf{X}_{m0}$ as the evolved point $p(m)$, in accordance with (2.5.8), and since $\mathbf{X}'_0$ is the point $p' = (P', -Q', R')$, $p' \in \mathsf{H}$ by definition, then

(6.6) $$p(m) = (P, -Q, R) = (P' - mx, -Q' - my, R' - mz),\ p \in \mathbb{Z}^3,\ (P', -Q', R') \in \mathsf{H}.$$

It is verified that this point $p(m)$ also satisfies the hyperbolic conservation equation (2.2.1) since, using (6.6),

(6.7) $$P^2 + Q^2 - R^2 = (P'^2 + Q'^2 - R'^2) + 2m(-xP + yQ + zR) + (x^2 + y^2 - z^2),$$

and, using equations (2.2.1) for p', (2.1.3) and (2.2.4), the right of (6.7) reduces to $+C^2$, i.e. (6.7) becomes

(6.8) $$P^2 + Q^2 - R^2 = +C^2.$$

This is just the original conservation equation (2.2.1) and, hence, $p(m)$ also satisfies (2.2.1) if p' satisfies (2.2.1), which it does by definition, therefore $p(m) \in \mathsf{H}$.

Moving on to the evolution of the $\mathbf{X}_-$ vector: this vector evolves according to (6.2c) and, at each positive increment in m $(m > 0)$, the vector changes by $\delta\mathbf{X}_{m-}$ as follows, and illustrated in Figure 4 below,

(6.9) $$\delta\mathbf{X}_{m-} = \mathbf{X}_{m-} - \mathbf{X}_{(m-1)-} = -(2m-1)\mathbf{X}_+ + 2\mathbf{X}'_0,\ m > 0.$$

It is seen that the change is similar to (6.5) but more rapid, as it grows by a factor $-(2m-1)$ of $\mathbf{X}_+$, i.e. an ever increasing lengthening, anti-parallel to $\mathbf{X}_+$, with only a slight compensatory constant change of $2\mathbf{X}_0'$. Thus, similar remarks for $\mathbf{X}_0$ apply to $\mathbf{X}_-$, and the angle with $\mathbf{X}_+$ also converges to 180 deg as m increases. For large m, $\mathbf{X}_-$ is, like $\mathbf{X}_0$, anti-parallel to $\mathbf{X}_+$, i.e. $\lim_{m\to\infty}\mathbf{X}_{m-} = -m^2\mathbf{X}_+$.

Also, as for $\mathbf{X}_0$, since changes in $\mathbf{X}_-$ are anti-parallel to $\mathbf{X}_+$ then, for example, if $\mathbf{X}_+ \in \mathbf{C}_\mathbf{U}$ then $\mathbf{X}_- \in \mathbf{C}_L$ and, conversely, if $\mathbf{X}_+ \in \mathbf{C}_\mathbf{L}$ then $\mathbf{X}_- \in \mathbf{C}_U$.

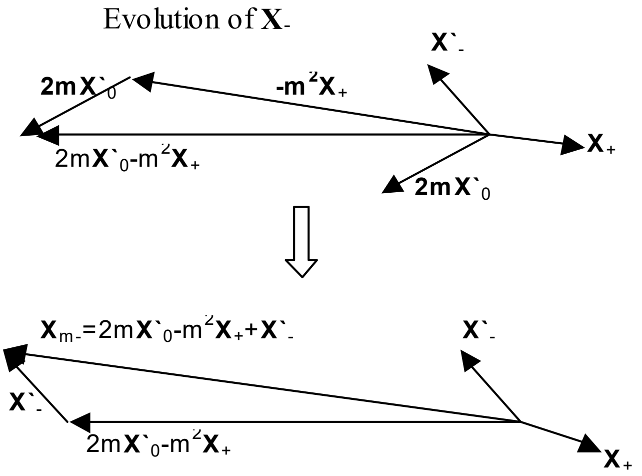

Figure 4 Evolution of X-

It is noted that the path of $\mathbf{X}_-$ only goes in one direction, on the opposite cone to that of the static $\mathbf{X}_+$, and, with the current sign convention chosen for m, never crosses the origin (strictly speaking, never traverses the x,y plane). The vector $\mathbf{X}_0$ also traces a unidirectional path in the same direction as $\mathbf{X}_-$, in the opposite direction to that pointed to by $\mathbf{X}_+$. Usually, but not exclusively, $\mathbf{X}_0$ also always points away from $\mathbf{X}_+$, i.e. $R<0$ for $\mathbf{X}_0$ as opposed to $z>0$ for $\mathbf{X}_+$, and so, it too, does not traverse the x,y plane. Note, however, it is possible for $\mathbf{X}_0$ to start ($m=0$) on the same side of the x,y plane as $\mathbf{X}_+$ in some cases, not discussed here. Whatever the case though, both $\mathbf{X}_0$ and $\mathbf{X}_-$ always traverse a unidirectional path and, when using the chosen sign convention $m>0$, the evolution is in the opposite direction to $\mathbf{X}_+$.

In summary, $\mathbf{X}_+$ is static, $\mathbf{X}_0$ grows linearly with constant vector $-\mathbf{X}_+$ added on each increment of m, and $\mathbf{X}_-$ grows quadratically, with an ever-increasing vector $-(2m-1)\mathbf{X}_+$, plus a small constant addition of $2\mathbf{X}_0$, added on each increment of m. Both $\mathbf{X}_0$ and $\mathbf{X}_-$ become anti-parallel with $\mathbf{X}_+$, in the large m limit, with $\mathbf{X}_-$ lying on the mirror image cone to $\mathbf{X}_+$, and $\mathbf{X}_0$ becoming ever closer, but never touching, the same mirror image cone in $\mathbf{C}$.

Since the sets $\mathbf{C}$ and $\mathbf{H}$ are disjoint, i.e. $\mathbf{C} \cap \mathbf{H} = \varnothing$ (4.10), the paths of $\mathbf{X}_-$ ($\mathbf{X}_- \in \mathbf{C}$) and $\mathbf{X}_0$ ($\mathbf{X}_0 \in \mathbf{H}$) never intersect, i.e. they never contain a common point.

This completes the discussion on eigenvector path evolution. The next section looks at the eigenvectors as a basis, and their highly oblique nature, as a prelude to studying the angular evolution of the eigenvectors and basic differential geometry.

(7) Eigenvectors as a basis

The eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ are suitable as a basis since they are linearly independent, as given by the following, non-zero vector triple product, where C is the positive, non-zero eigenvalue. An explanation of the derivation follows,

(7.1) $$\mathbf{X}_+ \wedge \mathbf{X}_0 \cdot \mathbf{X}_- = 2C^3 .$$

Using (2.1.2) for $\mathbf{X}_+$ and (2.1.6) for $\mathbf{X}_0$, the cross product in (7.1) is calculated as $\mathbf{X}_+ \wedge \mathbf{X}_0 = (Ry + Qz \quad -Rx + Pz \quad -Qx - Py)$. From the dynamical equations (2.2.3), this product reduces to $\mathbf{X}_+ \wedge \mathbf{X}_0 = C(x \quad y \quad -z)$, which is simply the conjugate vector $\mathbf{X}^-$ (2.3.1c), scaled by the eigenvalue C. Using (2.1.4) for $\mathbf{X}_-$, the full triple product now reduces to the inner product $\mathbf{X}^- \cdot \mathbf{X}_- = C(\alpha x + \beta y + \gamma z)$. By (2.2.7), the bracketed term $(\alpha x + \beta y + \gamma z) = 2C^2$, hence the result (7.1).

In fact, the vector triple product is a non-zero invariant, $+2C^3$, of the lattice, and independent of the value of the three arbitrary parameters k, l and m, Section (2.5).

Although linearly independent, the basis is far from orthonormal since none of the vectors has a unit magnitude and, in almost every case, they are not orthogonal to each other, except when R is zero – explained following. These properties can be seen from the following six, unique vector inner products, see also [2].

(7.2)

(7.2a) $$\mathbf{X}_+ \cdot \mathbf{X}_+ = \mathbf{X}^- \cdot \mathbf{X}^- = 2z^2$$

(7.2b) $$\mathbf{X}_+ \cdot \mathbf{X}_0 = \mathbf{X}^- \cdot \mathbf{X}^0 = 2zR$$

(7.2c) $$\mathbf{X}_+ \cdot \mathbf{X}_- = \mathbf{X}^- \cdot \mathbf{X}^{+^-} = 2(C^2 - \gamma z) = -2R^2 \text{, by (2.2.6c)}$$

(7.2d) $$\mathbf{X}_- \cdot \mathbf{X}_0 = \mathbf{X}^+ \cdot \mathbf{X}^0 = -2\gamma R$$

(7.2e) $$\mathbf{X}_- \cdot \mathbf{X}_- = \mathbf{X}^+ \cdot \mathbf{X}^+ = 2\gamma^2$$

(7.2f) $$\mathbf{X}_0 \cdot \mathbf{X}_0 = \mathbf{X}^0 \cdot \mathbf{X}^0 = C^2 + 2R^2$$

Firstly, there is a single, special case termed the 'almost trivial' solution, when $R = 0$, as documented in Appendix (C). In this case the mixed, eigenvector inner products, $\mathbf{X}_+ \cdot \mathbf{X}_0$ (7.2b), $\mathbf{X}_+ \cdot \mathbf{X}_-$ (7.2c) and $\mathbf{X}_- \cdot \mathbf{X}_0$ (7.2d), are all zero and, hence, $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ are orthogonal. Other than this exceptional case, z, γ, R and C can never be zero, by (5), and none of the above mixed products are consequently zero, hence the basis is not orthogonal. With the minimum absolute value of z, γ, R and C as unity, then neither are any of the mixed products $\mathbf{X}_+ \cdot \mathbf{X}_0$, $\mathbf{X}_+ \cdot \mathbf{X}_-$ and $\mathbf{X}_- \cdot \mathbf{X}_0$ unity and, consequently, none of the vectors is of unit magnitude. Of course, the basis can be diagonalised so the issue of non-othonormality is mainly one of academic interest. However, as an evolving triad of vectors, as discussed in the previous Section, the angles between them and how they evolve, in terms of the curvature of the path they trace in the lattice $\mathbf{L}$, is of interest and discussed next.

(8) Angles, Tangents and Flatness

The normal route to assess the geometry of a path in a vector space is by calculation of the tangent and normal vectors and application of the Serret-Frenet formulae, see [8]. This won't actually be done here, barring a quick look at the tangent vectors, because the basis vectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ are both discrete and distinctly non-orthonormal (previous section), and the vector algebra for the Serret-Frenet formulae is unnecessarily unwieldy. Instead, for the purposes of this paper, the evolution of curvature is obtained more simply by studying the scalar angles obtained from the inner products (7.2). The key properties are the same, irrespective of method, and the curvature of the $\mathbf{X}_-$ path will be seen to follow an inverse square law in parameter m.

Appendices (B) and (C) provide example data in the case of the simplest, non-trivial Pythagorean triple $(4,3,5)$ and the almost-trivial $(0,1,1)$. This data includes all eigenvectors and angles, given further below, plus data for 'flatness' parameter ω (8.13) and curvature κ (9.1) as they evolve.

A tangent vector $\mathbf{T}_0$, to the path of $\mathbf{X}_0$ in H, is calculated from (6.5) as follows, where $\delta m = 1$ since, trivially,

(8.0) $$\delta m = m - (m-1) = 1$$

(8.1) $$\mathbf{T}_0 = \frac{\delta \mathbf{X}_0}{\delta m} = -\mathbf{X}_+ .$$

Similarly, a tangent vector $\mathbf{T}_-$, to the path of $\mathbf{X}_-$ in C, is calculated from (6.9) as

(8.2) $$\mathbf{T}_- = \frac{\delta \mathbf{X}_-}{\delta m} = -(2m-1)\mathbf{X}_+ + 2\mathbf{X}'_0 .$$

Since $\mathbf{X}_+$ has no parametric variation with respect to m, its derivative, i.e. tangent vector, is zero and not considered further, except for completeness,

(8.3) $$\mathbf{T}_+ = \frac{\delta \mathbf{X}_+}{\delta m} = 0 .$$

From (8.1) it is seen that $\mathbf{T}_0$ is the constant vector $-\mathbf{X}_+$ and so there is no non-zero second derivative, and no intrinsic curvature of the $\mathbf{X}_0$ path as a consequence; it was mentioned in Section (6) that $\mathbf{X}_0$ traces a straight line parallel to $\mathbf{X}_+$.

(8.4) $$\frac{\delta \mathbf{T}_0}{\delta m} = 0 .$$

Richard J. Miller. Issue 2.0 June 2017
http://www.urmt.org

The $\mathbf{T}_-$ tangent vector (8.2) is not so trivial as it is dependent on m, and thus has a non-zero, second derivative and, therefore, some curvature.

It is this curvature that is investigated shortly, by examination of the evolving change in the scalar angle of $\mathbf{X}_-$ with respect to the static vector $\mathbf{X}_+$. Beforehand, however, the angles between all axes are examined.

Denoting the angle between $\mathbf{X}_+$ and $\mathbf{X}_0$ by θ_{+0}, the angle between $\mathbf{X}_+$ and $\mathbf{X}_-$ by θ_{+-}, and the angle between $\mathbf{X}_0$ and $\mathbf{X}_-$ by θ_{0-}, then θ_{+0}, θ_{+-} and θ_{0-} are obtained by the standard inner product relations, and depicted in Figure 5,

(8.5)

(8.5a) $\cos\theta_{+0} = \mathbf{X}_+ \cdot \mathbf{X}_0 / |\mathbf{X}_+||\mathbf{X}_0|$

(8.5b) $\cos\theta_{+-} = \mathbf{X}_+ \cdot \mathbf{X}_- / |\mathbf{X}_+||\mathbf{X}_-|$

(8.5c) $\cos\theta_{0-} = \mathbf{X}_0 \cdot \mathbf{X}_- / |\mathbf{X}_0||\mathbf{X}_-|$.

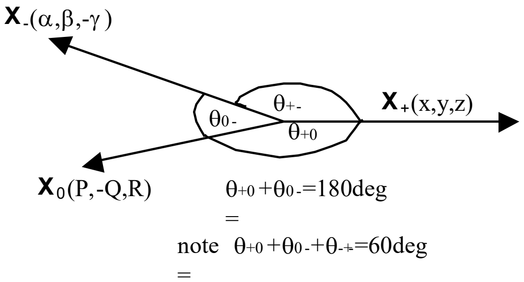

Figure 5 The Oblique Eigenvector Basis

The eigenvector magnitudes $|\mathbf{X}_+|$, $|\mathbf{X}_0|$ and $|\mathbf{X}_-|$ are calculated from definition (2.4.2) as follows: Using (2.1.2) for $\mathbf{X}_+$, then $\mathbf{X}_+ \cdot \mathbf{X}_+ = x^2 + y^2 + z^2$. Since $\mathbf{X}_+$ is a Pythagorean triple then, by (2.1.3), $\mathbf{X}_+ \cdot \mathbf{X}_+ = 2z^2$. Likewise, for $\mathbf{X}_-$ (2.1.4), by (2.1.5), $\mathbf{X}_- \cdot \mathbf{X}_- = 2\gamma^2$. Lastly, using (2.1.6) for $\mathbf{X}_0$, the inner product is $\mathbf{X}_0 \cdot \mathbf{X}_0 = P^2 + Q^2 + R^2$, and by (2.2.1), $P^2 + Q^2 = C^2 + R^2$, hence $\mathbf{X}_0 \cdot \mathbf{X}_0 = C^2 + 2R^2$. Taking the square root of these inner products gives the following expressions for the magnitudes

(8.6)

(8.6a) $|\mathbf{X}_+| = \sqrt{\mathbf{X}_+ \cdot \mathbf{X}_+} = \sqrt{2}z$

(8.6b) $|\mathbf{X}_0| = \sqrt{\mathbf{X}_0 \cdot \mathbf{X}_0} = \sqrt{C^2 + 2R^2}$

(8.6c) $|\mathbf{X}_-| = \sqrt{\mathbf{X}_- \cdot \mathbf{X}_-} = \sqrt{2}\gamma$

The mixed inner product expressions in the numerators of (8.5) are obtained from (7.2b) $\mathbf{X}_+ \cdot \mathbf{X}_0 = 2zR$, (7.2c), $\mathbf{X}_+ \cdot \mathbf{X}_- = -2R^2$ and (7.2d) $\mathbf{X}_- \cdot \mathbf{X}_0 = -2\gamma R$. Putting these expressions, plus (2.2.6c) and the magnitudes (8.6), into (8.5) gives the angles as follows, where $sg(zR)$ denotes the sign of term zR and $sg(\gamma R)$ the sign of term γR, see (8.7d), below, for the signum function 'sg' definition,

(8.7)

(8.7a) $$\cos\theta_{+0} = sg(zR)\left(2R^2/(C^2+2R^2)\right)^{1/2}$$

(8.7b) $$\cos\theta_{+-} = -R^2/(C^2+R^2)$$

(8.7c) $$\cos\theta_{0-} = -sg(\gamma R)\left(2R^2/(C^2+2R^2)\right)^{1/2}.$$

(8.7d) $$sg(a) = -1, a<0;\ sg(a)=0, a=0;\ sg(a)=+1, a>0$$

Since z and γ always have the same positive sign, by (2.5.2c) and (2.5.6c), the angles θ_{+0} and θ_{0-} are related as follows, using $\cos(180-\theta)+\cos(\theta)=0$, when restricting to the 0-180 deg range, which will be taken as the principle range throughout

(8.8) $$\theta_{+0} + \theta_{0-} = 180\,\text{deg}.$$

Note that since the eigenvectors are linearly independent, and therefore not co-planar, the sum of all three angles is not 360 deg, contrary to the appearance in Figure 5

(8.9) $$\theta_{+0} + \theta_{0-} + \theta_{+-} \neq 360\,\text{deg}.$$

Since $-R^2/(C^2+R^2)$ is always zero or less, then θ_{+-} in (8.7b) is always 90 deg or greater and, because $C>0$, it is always less than exactly 180 deg, hence θ_{+-} lies in the interval

(8.10) $$90 \le \theta_{+-} < 180\,\text{deg}.$$

The 90 deg equality is only satisfied for the almost trivial solution, when $R=0$, Appendix (C).

Restricting $\mathbf{X}_+$ to the upper cone such that $z>0$, $\mathbf{X}_+ \in \mathsf{C_U}$, then $\mathbf{X}_0 \in \mathsf{H_L}$ if $R \le 0$ (4.4) and, consequently, $sg(zR)$ in (8.7a) is minus, hence θ_{+0} also lies in the interval

(8.11) $$90 \le \theta_{+0} < 180\,\text{deg},\ z>0, R\le 0.$$

As an aside note, the condition $R \le 0$ is not guaranteed and it is possible for $\mathbf{X}_0$ to be such that $R>0$, i.e. $\mathbf{X}_0 \in \mathsf{H_U}$ (4.5). Nevertheless, given the sign convention adopted, $m>0$ (see Section (6)), then the solution always evolves quickly to $R \le 0$ when $\mathbf{X}_+$ lies in the upper cone $z>0$, $\mathbf{X}_+ \in \mathsf{C_U}$. So, for discussion purposes, this condition $R \le 0$ is safely adopted.

Given that $\mathbf{X}_0$ can never lie in a plane spanned by $\mathbf{X}_-$ and $\mathbf{X}_+$ (the vectors are linearly independent by (7.1)), then θ_{0-} can never be zero and, as per (8.11), restricting $\mathbf{X}_+$ to the upper cone gives

(8.12) $$0 < \theta_{0-} \le 90\deg\,,\ z > 0\,, R \le 0\,.$$

All three angles, θ_{+0}, θ_{+-} and θ_{0-} (8.7), are parameterised by the single variable R. With $\mathbf{X}_+$ static, i.e. k and l constant, then R is a function only of the evolution parameter m, as in (2.5.8c), and all angles therefore evolve with respect to m.

For the discussion of the angular evolution it is more useful, from both an algebraic and physical perspective, to re-parameterise equations (8.7) in terms of a single 'flatness' parameter ω, instead of R or m, and defined as follows.

Definition. The **flatness parameter** ω is defined as the ratio of the eigenvalue C to the dynamical variable R, for $R \ne 0$, (see below on transforming R away from zero),

(8.13) $$\omega = C / R\,,\ R \ne 0\,.$$

By dividing the angular equations (8.7) throughout by R^2, they can be neatly re-written, purely in terms of ω, as follows

(8.14)

(8.14a) $$\cos\theta_{+0} = +Sg(zR)\left(2/(2+\omega^2)\right)^{1/2}$$

(8.14b) $$\cos\theta_{+-} = -1/(1+\omega^2)$$

(8.14c) $$\cos\theta_{0-} = -Sg(\gamma R)\left(2/(2+\omega^2)\right)^{1/2}.$$

In this form, and disregarding the sign terms $Sg(zR)$ and $Sg(\gamma R)$, the angular equations are a function of ω^2 only.

Since R is parameterised by the evolution parameter m, as in $R = R' - mz$ (2.5.8c), then, whatever finite value for C is chosen, a value for m, and consequently R, can always be found such that $|R| >> C$ and $|\omega| << 1$ by (8.13), hence $\lim_{R\to\pm\infty} \omega = 0$. See also Section (14.8). Thus, the flatness parameter ω becomes ever smaller as the evolution progresses. As a reminder, the ability to modify R by choice of m, whilst keeping $\mathbf{X}_+$ and the eigenvalue C constant, is a consequence of the invariance principle in [1], and is known as an invariance transformation.

To obtain ω directly in terms of m, instead of R, substitute for R from (2.5.8c) into (8.13) to get

(8.15) $$\omega = C / (R' - mz).$$

For large m, and to first order in $1/m$, this approximates to

(8.16) $$\omega \approx \left(\frac{-C}{z}\right)\frac{1}{m}, \ |m| >> 1, \ |mz| >> |R'|.$$

Hence ω is inversely proportional to m, for large m, where C is a constant and z is constant by virtue of a static $\mathbf{X}_+$ when integer parameters k and l are constant.

Looking at θ_{+-} (8.14b) and θ_{+0} (8.14a), the choice of the name 'flatness' is now evident in that $\cos\theta_{+-}$ converges to –1 and $\cos\theta_{+0}$ also converges to -1 (for $z > 0$, $R < 0$), and so θ_{+-} and θ_{+0} both converge to 180 deg as $|\omega|$ becomes ever smaller, i.e. the axes flatten out. As noted earlier in Section (6), the $\mathbf{X}_-$ and $\mathbf{X}_0$ eigenvectors become parallel to each other and anti-parallel to $\mathbf{X}_+$:

(8.17) $$\lim_{R\to\pm\infty}\omega = \lim_{m\to\mp\infty}\omega = 0$$

(8.18) $$\lim_{\omega\to 0}\theta_{+-} = \lim_{\omega\to 0}\theta_{+0} = 180\,\text{deg}.$$

Rearranging (8.8) for θ_{0-} to get $\theta_{0-} = (180-\theta_{+0})\,\text{deg}$, and using (8.18), the limit $\lim_{\omega\to 0}\theta_{0-}$ is zero since

(8.19) $$\lim_{\omega\to 0}\theta_{0-} = 180 - \lim_{\omega\to 0}\theta_{+0} = 0\,\text{deg}.$$

It can easily be shown that θ_{+-} and θ_{+0} are never equal and, likewise, for θ_{+-} and θ_{0-}, as follows. By squaring and equating (8.14a) and (8.14b) a quadratic in ω^2 is obtained

(8.20) $$0 = 3\omega^2 + 2\omega^4.$$

This has no real solution for ω other than zero, which is not possible since C is never zero (2.2.1), and so ω in (8.13) is never zero. Hence θ_{+-} and θ_{+0} are never equal and, likewise, θ_{+-} and θ_{0-} are never equal since expressions θ_{+0} (8.14a) and θ_{0-} (8.14c) are equivalent, to within a sign.

Denoting the small angle $(\pi-\theta_{+-})rad$ by symbol $\bar{\theta}_{+-}$ as in

(8.21) $$\bar{\theta}_{+-} = (\pi-\theta_{+-})$$

then, using (8.14b), the cosine $\cos(\bar{\theta}_{+-})$ is given by

(8.22) $$\cos(\bar{\theta}_{+-}) = -\cos(\theta_{+-}) = +1\big/(1+\omega^2).$$

By expanding $\cos(\overline{\theta}_{+-})\deg$ as a power series (in radians), and the right-hand side of (8.22) as a power series in ω^2, the following two series are obtained, the second being convergent only for $|\omega| < 1$.

(8.23) $$\cos(\overline{\theta}_{+-}) = \left[1 - \frac{\overline{\theta}_{+-}^2}{2!} + \frac{\overline{\theta}_{+-}^4}{4!} - \frac{\overline{\theta}_{+-}^6}{6!} + \ldots\right]$$

(8.24) $$1/(1+\omega^2) = \left[1 - \omega^2 + \omega^4 - \omega^6 + \cdots\right],\ |\omega| < 1.$$

Comparing the two series, then ω is related to $\overline{\theta}_{+-}$, to second order, as follows:

(8.25) $$\omega \approx \frac{-\overline{\theta}_{+-}}{\sqrt{2}},\ O(\omega^2),\ |\omega| < 1.$$

The negative root is purposefully chosen since $\overline{\theta}_{+-}$ is always greater than zero, by (8.10) and (8.21), and ω is invariably less than zero, by its definition (8.13), since the values usually chosen for R (2.5.8c) make it negative, i.e. a particular solution R' in (2.5.8c) can always be made less than zero and, invariably, $m > 0$. It is stressed this is a choice, not a constraint, and it is really only $|\omega|$ that is of interest since its limiting value is zero anyhow.

Given there is no 3rd order term in (8.23) and (8.24), the approximation is better than second order, and the first error term is of the 4th order

(8.26) $$\left|\omega^4 - \overline{\theta}_{+-}^4\right| < \frac{\overline{\theta}_{+-}{}^4}{4},\ O(\omega^4),\ |\omega| < 1.$$

Thus, returning to $\overline{\theta}_{+-}$ (8.21), the flatness parameter ω (8.25) is approximately equal in magnitude to the angle $-(\pi - \theta_{+-})/\sqrt{2}$ for small ω, with the approximation becoming better as $\omega \to 0$

(8.28) $$\omega \approx -(\pi - \theta_{+-})/\sqrt{2},\ |\omega| << 1.$$

Once again, as in (8.25), the negative value has been chosen for ω since $\theta_{+-} < \pi$ and it is usual for ω to be negative, see above.

A relation between θ_{0-} and ω is obtained, for small ω, by squaring and dividing (8.14c) throughout by 2, and using the trig relation $2\cos^2\theta_{0-} = \cos(2\theta_{0-}) + 1$ to give

(8.29) $$\cos 2\theta_{0-} + 1 = 2\left(1 + \frac{\omega}{\sqrt{2}}\right)^{-1}.$$

Expanding both sides as power series for small ω and θ_{0-} gives

(8.30) $$\cos 2\theta_{0-} + 1 = 2 - 2\theta_{0-}^2 + \frac{2\theta_{0-}^4}{3} - \frac{(2\theta_{0-})^6}{6!} + \ldots$$

(8.31) $$2\left(1 + \frac{\omega}{\sqrt{2}}\right)^{-1} = 2 - \omega^2 + \frac{\omega^4}{2} - 2\left(\frac{\omega}{\sqrt{2}}\right)^6 + \ldots, \; |\omega| < 1 .$$

Comparing the two series, then ω is related to θ_{0-}, to second order, as follows

(8.32) $$\omega \approx -\sqrt{2}\theta_{0-}, \; O(\omega^2), \; |\omega| < 1$$

Given there is no 3rd order term, the approximation is better than second order, and the first error term is of the 4th order

(8.33) $$\left|\omega^4 - \theta_{0-}^4\right| = \frac{4\theta_{0-}{}^4}{3}, \; O(\omega^4), \; |\omega| < 1 .$$

Comparing (8.32) with (8.25) it is seen that θ_{0-} and $\overline{\theta}_{+-}$ are approximately related, for small ω, by

(8.34) $$\theta_{0-} \approx \frac{\overline{\theta}_{+-}}{2}, \; O(\omega^2), \; |\omega| < 1 .$$

The approximation getting better as ω becomes smaller such that

(8.35) $$\lim_{\omega \to 0} \theta_{0-} = \frac{\overline{\theta}_{+-}}{2} = \frac{(\pi - \theta_{+-})}{2} .$$

All this analysis is given some numbers in Appendices (B) and (C), and confirms the remarks on the size of large m, small ω, and the choice of signs in (8.25), (8.28) etc.

(9) Curvature

Curvature, symbol κ, is defined here as the rate of change of angle θ_{+-} with respect to the evolution parameter m, i.e.,

(9.1) $$\kappa = \delta\theta_{+-} / \delta m .$$

Changes in angle, $\delta\theta_{+-}$, on each evolutionary tick, δm, are given by the discrete difference $\delta\theta_{+-} = \theta_{(m)+-} - \theta_{(m-1)+-}$. With $\delta m = 1$, the curvature is thus defined as

(9.2) $$\kappa = \frac{\delta\theta_{+-}}{\delta m} = \theta_{(m)+-} - \theta_{(m-1)+-} .$$

Equations (8.16) and (8.28) are combined to obtain θ_{+-} in terms of m

(9.3) $$\theta_{+-} \approx \pi - \left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m}, \; |m| >> 1 .$$

And, applying (9.2), gives the curvature as follows

(9.4) $$\kappa \approx \left(\frac{-\sqrt{2}C}{z}\right)\left(\frac{1}{m} - \frac{1}{m-1}\right), \; |m| >> 1 .$$

For large m, this becomes

(9.5) $$\kappa \approx \left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m^2}, \; |m| >> 1 .$$

Note that, for large m, the discrete goes over to the continuum and this result could also be obtained by straightforward calculus on (9.3)

Thus, it is seen from (9.5) that the curvature of θ_{+-}, with respect to m, is an inverse square law. By (8.8) and (8.32), the same relation, barring a factor, applies to angles θ_{+0} and θ_{0-} respectively. Therefore all three angles θ_{+-}, θ_{0-} and θ_{+0} have an inverse square law curvature relation with respect to m, for large m.

Note that, once again, the expression for κ is a 'large m' approximation, becoming better as $|m|$ increases.

The curvature (9.5) is seen to be proportional to the eigenvalue C, which is effectively a free parameter for tuning; see Section (2.2).

If m is a time parameter, then the curvature is actually an angular rate with units rad/s. More usually, curvature is expressed by the Serret-Frenet formulae [8] with respect to arc-length, hence m would be a length parameter and the curvature is the rate of change of angle with distance along the path.

See table (B19) in Appendix (B) for a numeric example of κ.

Since $\omega \approx -(\pi - \theta_{+-})/\sqrt{2}$, by (8.28), then the rate of change of flatness $\frac{\delta\omega}{\delta m}$ is simply related to the curvature by

(9.6) $$\frac{\delta\omega}{\delta m} = \frac{1}{\sqrt{2}}\frac{\delta\theta_{+-}}{\delta m} = \frac{\kappa}{\sqrt{2}}, \; |m| >> 1.$$

Using (9.5) for κ this becomes

(9.7) $$\frac{\delta\omega}{\delta m} \approx \left(\frac{C}{z}\right)\frac{1}{m^2}, \; |m| >> 1.$$

Note that, with the chosen sign combinations, the curvature κ is positive, the flatness ω negative, and the rate of change of flatness $\frac{\delta\omega}{\delta m}$ (9.7) also positive, i.e. $\kappa > 0$, $\omega < 0$, $\frac{\delta\omega}{\delta m} > 0$. Regardless of sign, all limit to zero in the large m approximation.

(10) Mixed Angle Curvature

So far all reference has been made to 'flatness' and a flattening-out of the standard eigenvectors $\mathbf{X}_-$ and $\mathbf{X}_0$ with respect to $\mathbf{X}_+$, i.e. an alignment of eigenvectors as they evolve. However, it would be preferable that a basis formed from these eigenvectors would be orthogonal and not flat. As the theory here is three-dimensional, then an orthogonal triad is aesthetically appealing for physical comparison. To which end, there is nothing to stop the construction of such a basis from the standard eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$, given they linearly independent, see Section (7). The 'flattening' would then become more of a straightening whereby the axes fold out to a 90 deg angle with each other. It is not necessary to construct such a basis, at least in this paper, because the conjugate eigenvectors effectively play this role and can be illustrative in so far as angles are concerned. With that in mind, this section examines the angles between the standard and conjugate vectors, termed 'mixed angle curvature' herein, and these angles are seen to converge to a 90 deg angle as the flattening evolves. In fact, the convergence is seen to be even more rapid than the flattening (inverse cube rather than inverse square).

Denoting the angle between $\mathbf{X}_+$ and $\mathbf{X}^+$ by θ_+^+, the angle between $\mathbf{X}_-$ by $\mathbf{X}^-$ by θ_-^-, and the angle between $\mathbf{X}_0$ and $\mathbf{X}^0$ by θ_0^0, then θ_+^+, θ_-^- and θ_0^0 are obtained by the standard inner product relations

(10.1)

(10.1a) $$\cos\theta_+^+ = \mathbf{X}_+ \cdot \mathbf{X}^+ \big/ |\mathbf{X}_+|.|\mathbf{X}^+| = C^2/(C^2 + R^2)$$

(10.1b) $$\cos\theta_-^- = \mathbf{X}_- \cdot \mathbf{X}^- \big/ |\mathbf{X}_-|.|\mathbf{X}^-| = C^2/(C^2 + R^2)$$

(10.1c) $$\cos\theta_0^0 = \mathbf{X}_0 \cdot \mathbf{X}^0 \big/ |\mathbf{X}_0|.|\mathbf{X}^0| = C^2/(C^2 + 2R^2)\,.$$

The magnitudes in the above can be obtained from the standard eigenvector magnitudes (8.6) as follows

(10.2)

(10.2a) $$|\mathbf{X}^+| = |\mathbf{X}_-| = \sqrt{\mathbf{X}^+ \cdot \mathbf{X}^+} = \sqrt{2}\gamma$$

(10.2b) $$|\mathbf{X}^0| = |\mathbf{X}_0| = \sqrt{\mathbf{X}^0 \cdot \mathbf{X}^0} = \sqrt{C^2 + 2R^2}$$

(10.2c) $$|\mathbf{X}^-| = |\mathbf{X}_+| = \sqrt{\mathbf{X}^- \cdot \mathbf{X}^-} = \sqrt{2}z\,.$$

Note that the other mixed angles between $\mathbf{X}_+$ and $\mathbf{X}^0$, $\mathbf{X}_+$ and $\mathbf{X}^-$, $\mathbf{X}_-$ and $\mathbf{X}^0$, are all zero as a consequence of their conjugate, orthogonal definition (2.3).

Dividing the angular equations (10.1) throughout by R^2, they are re-written in terms of ω (8.13) as follows

(10.3)

(10.3a) $$\cos\theta_+^+ = \omega^2/(\omega^2 + 1)$$

(10.3b) $\cos\theta_-^- = \omega^2/(\omega^2+1)$

(10.3c) $\cos\theta_0^0 = \omega^2/(\omega^2+2)$.

As per (8.14), the angular equations (10.3) are only a function of the flatness parameter ω (8.13) in squared form but, unlike (8.14), they are unconditionally positive and have no usage of the Sg function.

From (10.3a) and (10.3b) it can be seen that, in the limit as $\omega \to 0$, the cosine of the angles θ_+^+ and θ_-^- tend to zero hence,

(10.4) $\lim_{\omega\to 0}\theta_+^+ = \lim_{\omega\to 0}\theta_-^- = 90\,\text{deg}.$

Given the angles θ_+^+ and θ_-^- converge to 90 deg then, to obtain small angle approximations, the angle $(\pi/2-\theta_+^+)\,rad$ is used instead and defined as angle $\overline{\theta}_+^+$ by

(10.5) $\overline{\theta}_+^+ = (\pi/2-\theta_+^+)$.

The expression (10.3a) is then given for $\overline{\theta}_+^+$ as

(10.6) $\sin\overline{\theta}_+^+ = \omega^2/(\omega^2+1)$.

By expanding $\sin(\overline{\theta}_+^+)$ as a power series (in radians), and the right of (10.6) as a power series in ω^2, the following two series are obtained, the second being convergent only for $|\omega|<1$.

(10.7) $\sin(\overline{\theta}_+^+) = \overline{\theta}_+^+ - \dfrac{\overline{\theta}_+^{+3}}{3!} + \ldots$

(10.8) $\omega^2/(1+\omega^2) = \omega^2 - \omega^4 + \cdots,\ |\omega|<1$.

Comparing the two series then, crudely seen, ω is related to $\overline{\theta}_+^+$, to first order in $\overline{\theta}_+^+$, as follows, where the negative root is taken, see the sign choice in (8.25).

(10.9) $\omega \approx -\sqrt{\overline{\theta}_+^+}\ ,\ O(\omega^2),\ |\overline{\theta}_+^+| << 1$.

Although certainly not a rigorous exercise in real-analysis, this approximation is seen to be numerically correct in Appendix (B), table (B21) and, as expected, the approximation improves as $\omega \to 0$.

Using the definition (10.5) for $\overline{\theta}_+^+$, the flatness parameter ω is approximately equal in magnitude to the square root of the angle $(\pi/2-\theta_+^+)$ for small ω

(10.10) $\omega \approx -\sqrt{(\pi/2-\theta_+^+)}$.

Since the expressions (10.3a) and (10.3b) for θ_-^- and θ_+^+ are identical, then (10.10) is duplicated for θ_-^- as

(10.11) $$\omega \approx -\sqrt{(\pi/2-\theta_-^-)}\,.$$

Lastly, for θ_0^0, its defining expression (10.3c) is almost identical to that of θ_+^+ (10.3a) and, by dividing throughout by a factor 2, it is re-written as follows

(10.12) $$\cos\theta_0^0 = (\omega/\sqrt{2})^2 \big/ ((\omega/\sqrt{2})^2+1)\,.$$

It is seen that replacing ω by $\omega/\sqrt{2}$ in (10.3a) makes it identical in form to (10.3c) for θ_+^+ and, hence, the result (10.9) can be directly translated for θ_0^0 as follows, where an extra $\sqrt{2}$ factor now appears on the right-hand side

(10.13) $$\omega \approx -\sqrt{2\bar{\theta}_0^0}\,,\ O(\omega^2)\,,\ \left|\bar{\theta}_0^0\right| << 1\,.$$

Equating ω (8.16) with (10.10) gives θ_+^+ in terms of m as

(10.14) $$\theta_+^+ \approx \pi/2 - \left(\frac{C^2}{z^2}\right)\frac{1}{m^2}\,,\ |m| >> 1\,.$$

Taking the derivative (finite difference) of θ_+^+ in (10.14) with respect to m, and making a large m approximation in the denominator, gives the following expression for the curvature $\kappa_+^+ = \dfrac{\delta\theta_+^+}{\delta m}$

(10.15) $$\kappa_+^+ \approx \frac{\delta\theta_+^+}{\delta m} = \left(\frac{2C^2}{z^2}\right)\frac{1}{m^3}\,, |m| >> 1\,.$$

It is seen that the curvature κ_+^+ is an inverse cubic function of m, as opposed to an inverse square law for θ_{+-} curvature κ, as in (9.5). Re-writing (10.15) in terms of κ, and disregarding the proportionality constant, gives

(10.16) $$\kappa_+^+ \propto \frac{\kappa}{m}\,, |m| >> 1\,.$$

Comparing (9.5) with (10.16), it is seen that the angular convergence (to 90 deg), between $\mathbf{X}_+$ and $\mathbf{X}^+$ (θ_+^+), is more rapid than the flattening between $\mathbf{X}_+$ and $\mathbf{X}_-$ (θ_{+-}). Otherwise, similar remarks apply. See table (B22) in Appendix (B) for a numeric example of κ_+^+.

- End of Part I -

- Part I I-

(11) Physical Considerations Overview

Fundamentally, the matrix $\mathbf{A}$ and its associated eigenvectors, $\mathbf{X}_+$, $\mathbf{X}_0$, and $\mathbf{X}_-$, arose from an invariance principle that was applied algebraically as local and global transformations, to dynamical variables in a conservation equation in [1]. This was purely and simply mathematical physics, but with a couple of twists: 1) the conservation equation was generally an abstraction, albeit with some consideration of energy conservation; 2) the development was entirely in integers and more the realm of number theoretic issues such as power residues, primitive roots, and nth order Diophantine equations.

Overall, the work is considered to be physics in integers and, as such, this second part of the paper is intended to highlight some plausible connections with mathematical physics and the physical world. It is only intended as a tentative introduction but, nonetheless, is meant to highlight what the author considers are connections too interesting to ignore.

Association of all equations and variables to physical quantities requires dimensional consistency if they are to represent the physical world. As such, a dimensional analysis is given before physical associations are made.

(12) Dimensional (Units) Analysis

By examination of equations (2.1.3), (2.1.5) and (2.2.1), it is clear that

(12.1) x, y, z, $\mathbf{X}_+$, $\mathbf{X}^-$ all have the same units,

(12.2) α, β, γ, $\mathbf{X}_-$, $\mathbf{X}^+$ all have the same units,

(12.3) P, Q, R, C, $\mathbf{X}_0$, $\mathbf{X}^0$ all have the same units.

A 'units' function is defined as follows

(12.4) $units(mass) = M$
(12.5) $units(length) = L$
(12.6) $units(time) = T$
(12.7) $units(m) =$ to be determined.

The units of each set of triples will be related via the evolution parameter m, which will be left undefined for the moment (12.7). Whatever the units of m are, the units of (12.1) to (12.3) are determined as follows, where the standard eigenvectors, $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$, are used to represent all the quantities in (12.1) to (12.3), excepting eigenvalue C.

By examination of (2.5.8), the units of $\mathbf{X}_+$ are related to $\mathbf{X}_0$ by

(12.8) $$units(\mathbf{X}_+) = \frac{units(\mathbf{X}_0)}{units(m)}.$$

Examination of (2.2.7) shows that

(12.9) $$units(\mathbf{X}_-) \times units(\mathbf{X}_+) = units(\mathbf{X}_0)^2$$

and, using (12.8), this gives for the units of $(\mathbf{X}_-)$

(12.10) $$units(\mathbf{X}_-) = units(\mathbf{X}_0) \times units(m).$$

Lastly, the units of integer parameters k and l in (2.5.1) are, from (2.5.2),

(12.11) $$units(k,l) = \sqrt{units(\mathbf{X}_+)}.$$

and, similarly, the units of integer parameters s and t in (2.5.4) are

(12.12) $$units(s,t) = \sqrt{units(\mathbf{X}_-)}.$$

It is seen from (12.8) that, whatever the units of $P, Q, R, C, \mathbf{X}_0, \mathbf{X}^0$, those of $x, y, z, \mathbf{X}_+, \mathbf{X}^-$ are equivalent to the derivative of $P, Q, R, C, \mathbf{X}_0, \mathbf{X}^0$ with respect to m, i.e. division by m, and those of $\alpha, \beta, \gamma, \mathbf{X}_-, \mathbf{X}^+$ are the integral, i.e. multiplication by m.

Using the standard calculus derivative $\frac{d}{dm}$ as a good, large m approximation for discrete differences, i.e.

(12.13) $$\frac{d}{dm} \approx \frac{\delta}{\delta m}, \; m >> 0, \; \delta m = 1,$$

then the above dimensional relations can be verified by looking at the derivatives of the evolution equations (6.1), as follows,

(12.14) $$units\left(\frac{d\mathbf{X}_0}{dm}\right) = units(\mathbf{X}_+)$$

(12.15) $$units\left(\frac{d\mathbf{X}_-}{dm}\right) = units(\mathbf{X}_0).$$

Differentiating (12.15) a second time, and using (12.14), gives

(12.16) $$units\left(\frac{d^2\mathbf{X}_-}{dm^2}\right) = units\left(\frac{d\mathbf{X}_0}{dm}\right) = units(\mathbf{X}_+).$$

The derivative $\frac{d\mathbf{X}_{m0}}{dm}$ is calculated from (6.2b) as follows, confirming (12.14)

(12.17) $$\frac{d\mathbf{X}_{m0}}{dm} = -\mathbf{X}_+.$$

Likewise, the derivative $\frac{d\mathbf{X}_{m-}}{dm}$ is calculated from (6.2c) as follows, confirming (12.15), since the term $-2m\mathbf{X}_+$ has the same units as $\mathbf{X}_0$, by (12.8),

(12.18) $$\frac{d\mathbf{X}_{m-}}{dm} = -2m\mathbf{X}_+ + 2\mathbf{X}_0'.$$

Taking the derivative of (12.18) to get the second order derivative $\frac{d^2\mathbf{X}_{m-}}{dm^2}$ confirms (12.16) since

(12.19) $$\frac{d^2\mathbf{X}_{m-}}{dm^2} = -2\mathbf{X}_+$$

It is also seen to be constant since $\mathbf{X}_+$ is static, by (6.2a), formalised as

(12.20) $$\frac{d\mathbf{X}_+}{dm} = 0$$

For the same reason, the second order derivative $\frac{d^2\mathbf{X}_{m0}}{dm^2}$ is zero by (12.17)

(12.21) $$\frac{d^2\mathbf{X}_{m0}}{dm^2} = 0$$

These derivative relations will be seen to be important in relating the variables to physical quantities in the next Section. See also Section (14.5).

(13) Physical Associations

This Section makes some tentative associations of all the variables x, y, z, P, Q, R, C and α, β, γ to real-world quantities.

Given that (2.2.1) is a conservation equation with sum-squared quantities, the natural assumption is to attribute the squared terms to kinetic energies and, hence, associate the dynamical variables P, Q, R to momentum per unit mass, i.e. velocity. This was alluded to in [1] and thus their suggestive name, 'dynamical variables', albeit that name was primarily motivated by the invariance principle (likened to invariance of momentum to translations). Whilst these quantities are linear, they could equally be angular, i.e. angular velocity and momentum. Indeed, [1] makes the comparison of the conservation equation and invariance principle with that of angular rather than linear momentum. Nevertheless, for familiarity, linear quantities will be used for the following discussion. Lastly, these comparisons are tentative and the exact physical nature of this work is yet to be determined [2017_3].

Given the arguments in Section (12), and since force is the derivative of momentum with respect to time, then the following physical associations are made,

(13.2)
(13.2a) m ~ time
(13.2b) $\mathbf{X}_0$, P, Q, R, C ~ momentum
(13.2c) $\mathbf{X}_+$, x, y, z ~ force (momentum per unit time)
(13.2d) $\mathbf{X}_-$, α, β, γ ~ momentum$\times$time (mass$\times$length).

The last association of $\mathbf{X}_-$ with momentum$\times$time does not seem to have a physical association except when all three quantities are considered as 'per unit mass', then $\mathbf{X}_-$ is equivalent to position, momentum $\mathbf{X}_0$ is equivalent to velocity, and force $\mathbf{X}_+$ equivalent to acceleration, i.e.

(13.3)
(13.3a) $\mathbf{X}_0$, P, Q, R, C ~ velocity (momentum per unit mass)
(13.3b) $\mathbf{X}_+$, x, y, z ~ acceleration (rate of change of momentum per unit mass)
(13.3c) $\mathbf{X}_-$, α, β, γ ~ position (momentum per unit mass x time).

Given that $\mathbf{X}_+$ is static, i.e. a constant vector (6.2a), then (13.2c) represents a constant force (per unit mass), and (13.3b) a constant acceleration (per unit mass), i.e.

(13.4) $$\frac{d\mathbf{X}_+}{dm} = 0$$ ~ constant force = constant acceleration per unit mass

The derivative relations (12.17) to (12.19) are also consistent with the interpretations (13.2) and (13.3), given as follows, all per unit mass:

(13.5) $$\frac{d\mathbf{X}_{m0}}{dm} = -\mathbf{X}_+ \sim \text{ rate of change of velocity = acceleration}$$

(13.6) $$\frac{d\mathbf{X}_{m-}}{dm} = -2m\mathbf{X}_+ + 2\mathbf{X}'_0 \sim \text{rate of change of position = velocity}$$

(13.7) $$\frac{d^2\mathbf{X}_{m-}}{dm^2} = -2\mathbf{X}_+ \sim \text{rate of change of velocity = acceleration.}$$

Integrating (13.5) gives $\mathbf{X}_{m0} = -m\mathbf{X}_+ + \mathbf{X}'_0$, as per (6.2b), i.e. the velocity $\mathbf{X}_{m0}$ starts with an initial value $\mathbf{X}'_0$ and increases linearly in magnitude with time, i.e. constant acceleration $\mathbf{X}_+$.

Likewise, integrating the acceleration of $\mathbf{X}_{m-}$ (13.7) once gives its velocity as $-2m\mathbf{X}_+ + 2\mathbf{X}'_0$ (13.6), which is actually twice the velocity of $\mathbf{X}_{m0}$ (6.2b). Integrating a second time returns the position $\mathbf{X}_{m-}$ as per (6.2c).

Comparing (13.5) and (13.7), the position $\mathbf{X}_{m-}$ accelerates with an acceleration $-2\mathbf{X}_+$, twice that of $\mathbf{X}_{m0}$ but in the same direction, and so the velocity $\mathbf{X}_{m0}$ (6.2b) is always half that of $\mathbf{X}_{m-}$ (13.6).

In terms of cones and hyperboloids, Section (4), these associations are interpreted as follows: with a static vector $\mathbf{X}_+$ in the upper cone, i.e. $\mathbf{X}_+ \in \mathbf{C}_U$, then the $\mathbf{X}_0$ vector accelerates at a constant rate $-\mathbf{X}_+$, always in the opposite direction to $\mathbf{X}_+$ and, consequently, along the surface of the lower hyperboloid $\mathbf{H}_L$. Simultaneously, the $\mathbf{X}_-$ vector also accelerates at a constant rate, $-2\mathbf{X}_+$ along the surface of the lower cone $\mathbf{C}_L$, forever going twice the speed of $\mathbf{X}_0$.

The evolution parameter m could also be associated with inverse time or inverse length or, indeed, inverse other, instead of time. Looking at dimensional equations (12.8) to (12.10), this would then make $\mathbf{X}_+$ a position vector which, given its elements are labelled with the traditional Cartesian x, y, z, intuitively makes sense. The vector $\mathbf{X}_0$ would remain a velocity and the $\mathbf{X}_-$ vector would then be acceleration. Superficially then, this would seem to be a simple re-labelling exercise or swap in $\mathbf{X}_+$ and $\mathbf{X}_-$, which it is, and is closely related to a duality between $\mathbf{X}_+$ and $\mathbf{X}_-$, i.e., denoting a dual variable by an over-struck tilde, then $\mathbf{X}_- = \widetilde{\mathbf{X}}_+$ or $\mathbf{X}_+ = \widetilde{\mathbf{X}}_-$. See also Section (14.9).

(14) Concepts in Mathematical Physics

This last Section is a short collection of some key links with mathematical physics and the physical world. The list is both speculative in places and far from exhaustive, with some key points still under consideration and others omitted since they are the subject of ongoing work for future publication. [2017_4]

(14.1) Quantisation. First and foremost, all the work here and in [1] and [2] is entirely in integers, and is therefore quantised from the beginning. The development has been pursued explicitly avoiding rational, real or complex numbers. Nevertheless, complex numbers have their isomorphs in the form of integer, unity roots (or primitive roots and power residues), which are the dynamical variables P,Q,R. With this isomorphism in mind, trace-free, complex matrices, used throughout mathematical physics, are not so dissimilar to matrix $\mathbf{A}$ (1.1). [2017_5]

(14.2) No Singularities. Section (5) explains that the zero point is not possible as an element of the lattice, i.e. neither the cone $\mathbf{C}$ (4.3) nor hyperboloid $\mathbf{H}$ (4.6), contain the origin $(0,0,0)$. This is fundamentally due to the fact that, for $\mathbf{C}$, the eigenvalue C is non-zero - a separate eigenvalue $\lambda = 0$ is already allocated for eigenvectors $\mathbf{X}_0, \mathbf{X}^0 \in \mathbf{H}$; for $\mathbf{H}$, $(P,Q,R) = (0,0,0)$ is never a valid solution to the hyperbolic equation (2.2.1). Hence $\mathbf{C}$ and $\mathbf{H}$ are referred to as having no singularity. As $\mathbf{H}$ is a discrete quadric surface, its real $\mathbb{R}^3$ analogue would, anyhow, ordinarily have a finite radius (eigenvalue C) in the x,y plane. On the other hand, $\mathbf{C}$ comprises both upper and lower cones, $\mathbf{C}_U$ and $\mathbf{C}_L$, and its $\mathbb{R}^3$ analogue always includes the origin where the tips meet. But, when working in $\mathbb{Z}^3$, as stated in Section (5), this is no longer possible. As a consequence, any path connecting the cones would skip the origin when going from $\mathbf{C}_L$ to $\mathbf{C}_U$ (evolving forward) and vice versa, $\mathbf{C}_U$ to $\mathbf{C}_L$ (evolving backward); likewise for a path on the hyperboloid $\mathbf{H}$.

(14.3) No Infinities. It is noted in [1] that zero divisors are possible but can be removed by transformation without altering $\mathbf{X}_+$. By definition, this transforms $\mathbf{X}_0$ to remove the zero, and $\mathbf{X}_-$ transforms as a consequence, maintaining invariance in (2.2.5) – see also (14-7) further below. This transformation property has not been explicitly employed herein but is mentioned in connection with singularities as a useful property for any awkward expressions, in particular the indeterminate form $0/0$, which can arise in [1].

(14.4) Symmetry. The simplifying 'Pythagoras Conditions' in [1], that reduce the unity root matrix theory in [2] to that of Pythagoras, represent a transition from an asymmetric to symmetric set of equations and solutions. The Pythagoras 'state' actually represents a very symmetric, zero Potential energy form. However, this is really the realm of extensions to [1] and mentioned here only as further evidence of links to the physical world. The local and global invariance transformations, their affect on symmetry and, in particular, the vanishing Potential term V in the conservation equation, are considered analogous to gauge transformations in field theory. Note that the Potential term is not shown in (2.2.1) precisely because it is zero for Pythagoras.

(14.5) Calculus. The physical associations in Section (13) compare the eigenvectors $\mathbf{X}_-$, $\mathbf{X}_0$ and $\mathbf{X}_+$ to a position, velocity and acceleration vector respectively, and notes that they could equally well be angular equivalents or other unspecified. Whatever the association, the important point is that they span constant, first and second order vector derivatives with respect to an evolutionary parameter m (~time), as shown in the dimensional analysis. Furthermore, the second order derivative (~acceleration) is constant and so there are no higher-order, non-zero derivatives.

(14.6) A Conserved Non-zero, Zero-point Energy. The quantity C^2 in equation (1.1) is analogous to energy in Ref. [1] and split into a kinetic and potential term, albeit an abstraction. Nevertheless, since it is the square of a non-zero eigenvalue, it is never zero, and its smallest value, $C^2 = +1$, is akin to a 'zero-point' energy, per unit mass. The zero-point energy is always non-zero and, for a single oscillator, is given by $E_0 = \hbar\omega/2$ for oscillator frequency ω. Using the Planck frequency (the reciprocal of the Planck time, see (14.8) below), this gives the rather large energy $E \approx 10^{19} GeV$ ($2\times 10^9 J$). The impact of a non-zero C (or C^2) is wider reaching since it also dictates that a trivial, zero Pythagorean triple $(0,0,0)$ is impossible within the theory, as explained in Section (5), and discussed above in (14.2).

(14.7) Invariants. The three eigenvalues $\lambda = \pm C,0$ are, by definition, invariants of the theory. The eigenvector space generates six other scalar invariants via the vector, inner product relations between the three eigenvectors and their conjugate forms. Of course, three of these are zero by the orthogonal properties between row and column eigenvectors vectors with distinct eigenvalues. The full suite of inner products is a set of six equations, given earlier in this paper and reproduced below,

(2.1.3) $\mathbf{X}^-\mathbf{X}_+ = x^2 + y^2 - z^2 = 0$, Pythagoras equation

(2.1.5) $\mathbf{X}^+\mathbf{X}_- = \alpha^2 + \beta^2 - \gamma^2 = 0$, Pythagoras equation

(2.2.1) $\mathbf{X}^0\mathbf{X}_0 = P^2 + Q^2 - R^2 = C^2$, Dynamical conservation equation

(2.2.7) $\mathbf{X}^+\mathbf{X}_+ = \mathbf{X}^-\mathbf{X}_- = \alpha x + \beta y + \gamma z = 2C^2$, Potential equation

(2.2.4) $\mathbf{X}^0\mathbf{X}_+ = \mathbf{X}^-\mathbf{X}_0 = xP - yQ - zR = 0$

(2.2.5) $\mathbf{X}^0\mathbf{X}_- = \mathbf{X}^+\mathbf{X}_0 = \alpha P - \beta Q + \gamma R = 0$.

The volume element gives another, derived invariant $2C^3$

(7.1) $\mathbf{X}_+ \wedge \mathbf{X}_0 \cdot \mathbf{X}_- = 2C^3$.

The important point about these values is that, for any evolved set of eigenvectors $\{\mathbf{X}_+, \mathbf{X}_{m0}, \mathbf{X}_{m-}\}$ and their conjugates, they are truly invariant in the lattice **L** (4.7). The invariants cover the integer set $\{-C, 0, C, C^2, 2C^2, 2C^3\}$, and it is noticed that, for unity C, this set covers the most basic integers $\{-1,0,1,2\}$. Even when $C \neq 1$, their ratios also include the simple set of integers $\{0, \pm\frac{1}{2}, \pm 1, \pm 2\}$. Given the musings on angular momentum conservation in [1], it is tempting, but admittedly extremely tenuous, to think of spin [2017_6].

(14.8) Scale As the lattice **L** (4.7) is discrete, by definition, it is evident that at some stage it must appear continuous in the macroscopic world. Difference equations must become differential equations and, given almost the entire world of mathematical physics works with continuous differential equations (even string theory), the integer values used herein must be very large such that a numeric difference of 1 or 2 is relatively tiny, and all quantities appear continuous.

A tentative scale can be obtained by looking at the smallest possible interval of evolution, i.e. $m = 1$. At the smallest scale of physical reality, considered to be either the Planck time or length, a value of $m = 1$ represents a length of 1.6×10^{-35} m or a time of 5.4×10^{-44} s, see Penrose [7], $t = \sqrt{(\bar{h} G c^{-5})}$ and $l = \sqrt{(\bar{h} G c^{-3})}$, $\bar{h} = 1.055 \times 10^{-34} Js(Kgm^2 s^{-1})$, $G = 6.673 \times 10^{-11} Kg^{-1} m^3 s^{-2}$, $c = 2.998 \times 10^8 ms^{-1}$. So, at the one metre length, m is $\approx 10^{35}$ and, for a time of 1s, m is $\approx 10^{43}$. Thus, there is no issue with approximating the continuous by the discrete with these sizes of numbers.

The smallest magnitude solution for a Pythagorean triple $\mathbf{X}_+$ or $\mathbf{X}_-$ is $(0,1,1)$, according to definition (1.0). This is the 'almost trivial' solution, see Appendix (C), and has an associated, smallest point in **H**, eigenvector $\mathbf{X}_0 = (C,0,0)$, i.e. its magnitude is the eigenvalue C. The level represented by the 'position' eigenvector $\mathbf{X}_- = (0,1,-1)^T$, with magnitude $\sqrt{2}$, is thus considered the Planck level, i.e. if it represented distance it would be around 10^{-35} m.

When making physical comparisons on the flatness ω (8.16) and curvature κ (9.5) of the lattice, the question arises, how large does 'large m' have to be before the granularity of a discrete lattice starts to appear continuous and the flatness ω becomes imperceptibly zero? The short answer is not very large, but it is dependent on the value chosen for the conserved quantity given by eigenvalue C. This can be most obviously seen in (8.16) where ω is proportional to C and inversely proportional to m, for constant z. As noted in (14.6), given that C can also be large but finite, a value for m can always be chosen to make ω as small as desired. Indeed, Ref. [1] chose $C = 1$, as in equation (2.1.1), and with this value a 'large m' can be as small as $m = 10$. The flatness is inversely proportional to m, as given by (8.16), and the curvature (9.5) is inversely proportional to the square of m. Considering m as units of Planck time, then a value $m = 10$ represents $\approx 10^{-42} s$. This short evolutionary period is discussed again below. Nevertheless, whatever the scaling of m, it is clear that a large m does not have to be very large before the angles θ_{+-} and θ_{0-} become very close to 180 deg, and the flatness ω all but zero, i.e. vectors $\mathbf{X}_0$ and $\mathbf{X}_-$ align anti-parallel with $\mathbf{X}_+$.

The above discussion on ω and κ assumes z is constant in (8.16) but, since this is the z component in eigenvector $\mathbf{X}_+$, it can be made as large as desired by suitable choice of k and/or l (2.5.2c), to nullify the effect of a large C. Given $\mathbf{X}_+$ is static then, once chosen, the evolution proceeds as per a small C. This freedom to vary z is because $\mathbf{X}_+$ is a Pythagorean triple and the theory herein is valid for all Pythagorean triples, $z > 0$; noting that [2] actually also extends to all Pythagorean triples such that $z < 0$. Albeit this is not necessary since the region $z < 0$ is effectively the conjugate world of $\mathbf{X}^+$ and arguments of duality apply, i.e. one can either work in the standard or dual vector space, but working in both is superfluous.

Translating this discussion on scale to a cosmological analogy then, given that m need not be very large at all, i.e. $m = O(10^1)$, on an evolutionary scale of $m = O(10^{43})$, the flattening is all but over in the very early, inflationary stages (first 10 ticks). Certainly by 10^{43} ticks (1s) flatness reigns supreme. With a flattening period (a function of the clock ticks m) tamed by choice of a large starting energy C^2 then, from (8.16), it is clear that the larger the starting energy (for a fixed z, see above), the longer the evolutionary period required to attain flatness.

Associating eigenvalue C with speed (13.3a), and C^2 as energy per unit mass (kinetic energy/mass), then the Planck unit of C is simply the speed of light, little c, i.e. $C = c \approx 3\times10^8 m/s$. Given the age of the universe is approximately 13 billion years, which equates to about 10^{17} seconds, then the evolution parameter is, in units of Planck time, $m \approx 10^{62}$. Using the definition of the flatness parameter $\omega \approx \left(\frac{-C}{z}\right)\frac{1}{m}$ (8.16), for large m, with $z = 1ms^{-2}$ as in the 'almost trivial' solution, Appendix (C), and $C \approx 3\times10^8 m/s$, then the flatness is around $\omega \approx 10^{-54}$ (dimensionless), i.e. flat to within 1 part in 10^{54}.

(14.9) Scale Duality

Although the topic of a duality ($\mathbf{X}_- = \tilde{\mathbf{X}}_+$), mentioned at the end of Section (13), is beyond the scope of this paper, as a prelude, the following observation is supplied. If m is dimensionless, then all three eigenvectors have the same units. In such a case, the evolution equations (6.2) show that, in the large m limit, to within an arbitrary choice of sign, $\mathbf{X}_{m-}$ tends to $m^2\mathbf{X}_+$. Thus, the large m formulation in $\mathbf{X}_-$ is simply a scale factor m^2 of the formulation in $\mathbf{X}_+$ and, hence, $\mathbf{X}_{m-}$ is considered the dual of $\mathbf{X}_+$.

In terms of the null-cone sets $\mathbf{C}_L$ and $\mathbf{C}_U$ then, since $\mathbf{X}_- \in \mathbf{C}_L$ when $\mathbf{X}_+ \in \mathbf{C}_U$, (Section (14)), this represents a duality between the small and large-scale geometry of the sets expressed as $\mathbf{C}_L = \tilde{\mathbf{C}}_U$, or $\mathbf{C}_U = \tilde{\mathbf{C}}_L$. With the middle ground (macroscopic world) considered to be that of the eigenvector $\mathbf{X}_{m0}$ (residing in the disjoint, hyperbolic set $\mathbf{H}$) then, relative to $\mathbf{X}_{m0}$, the microscopic region is $\mathbf{X}_+$ and the large scale region that of $\mathbf{X}_{m-}$. Using (6.2), when viewed with respect to $\mathbf{X}_{m0}$, the vector $\mathbf{X}_{m+}$ tends to $\frac{1}{m}\mathbf{X}_+$ and $\mathbf{X}_{m-}$ tends to $m\mathbf{X}_+$ for large m, i.e. $\mathbf{X}_{m0}$ sees an $\frac{1}{m}, m$ duality between the microscopic and the very large.

The $\frac{1}{m}, m$ duality is analogous to mirror manifold symmetry in modern mathematical physics, see the subject of mirror symmetries in string theory, Section (31.14) in [7]. It is also of note that the parameter m is analogous to a 'winding number' since it controls the quotient in a moduli relation. For example, using $P = P' - mx$ (2.5.8), the dynamical variable P has the congruence property $P \equiv P'(\bmod x)$ by virtue of its definition as a power residue $P^2 \equiv C^2 \pmod{x}$, (A7a); when $C = +1$ the dynamical variable P is termed a unity root. The quotient in (2.5.8) is the evolutionary parameter m and is the equivalent of the winding number.

(14.10) Minkowski Geometry. Not coincidentally, since the points in **C** satisfy the Pythagoras equation (2.1.3) and (2.1.5), **C** is a discrete, 3D version of the null light cone in the 4D Minkowski space of the Special Theory of Relativity (STR). Hence **C** is also referred to here as the discrete, null cone as its vectors, like STR, also have zero norm (length), see (2.4.1). Reference [7] gives a good, popular account with two chapters on Spacetime and Minkowskian Geometry. A 4D version of the work in this paper is pending and no further comment is added at this stage [2017_7].

(15) Summary

The paper started with a review of its predecessor [2], itself a simplification deriving from [1]. Ref. [2] showed that the $\mathbf{A}$ matrix (1.1) has three eigenvectors $\mathbf{X}_+$ (2.1.2), $\mathbf{X}_0$ (2.1.6) and $\mathbf{X}_-$ (2.1.4), for eigenvalues $\lambda = +1$, $\lambda = 0$ and $\lambda = -1$ respectively. The theory is extended in this paper to cover the more general eigenvalues, $\lambda = +C, 0, -C$, $C \in \mathbb{Z}$, $C > 0$. The components of the two eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ satisfy the Pythagoras equations (2.1.3) and (2.1.5) respectively, whilst the third eigenvector satisfies the hyperbolic, dynamical conservation equation (2.2.1).

A full parametric solution for the eigenvectors is given in Section (2.5), and it is seen that fixing two of the arbitrary integer parameters, k and l (2.5.1), fixes $\mathbf{X}_+$, and enables $\mathbf{X}_0$ and $\mathbf{X}_-$ to be related to $\mathbf{X}_+$ by a third, arbitrary integer parameter m. This parameter is identified as an evolution parameter, which relates the evolution of $\mathbf{X}_0$ and $\mathbf{X}_-$ with respect to $\mathbf{X}_+$.

Since all equations and variables are integers only, and with three arbitrary integer parameters, the eigenvectors represent points in a discrete lattice L in $\mathbb{Z}^3$ (4.7). The lattice comprises two discrete cones, C_U (4.1) and C_L (4.2), collectively covered by the set C (4.3), in which the eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ reside, and a discrete hyperboloid H (4.6), the home of $\mathbf{X}_0$.

The paper then proceeds to give a simple, geometric interpretation of the eigenvectors in terms of their evolving path in L. Considering $\mathbf{X}_+$ as a fixed eigenvector in the upper cone C_U, $z > 0$, it is shown that the $\mathbf{X}_-$ evolution (6.2c), with respect to parameter m, is a null path on the lower cone C_L, $-\gamma < 0$, tracing a curved, downward path and always pointing away from $\mathbf{X}_+$, becoming anti-parallel to $\mathbf{X}_+$. The curvature of the $\mathbf{X}_-$ path is shown to follow an inverse square law in m, for large m. The eigenvector $\mathbf{X}_0$ (6.2b) similarly evolves by tracing a downward path on the lower hyperboloid H_L (4.4), albeit following a straight line, anti-parallel to $\mathbf{X}_+$, with a slower growth and linear in m, as opposed to $\mathbf{X}_-$, which grows quadratically with m.

As a basis, the three eigenvectors were found to be highly oblique and become ever more oblique as they evolve, 'flattening' out such that $\mathbf{X}_-$ and $\mathbf{X}_0$ become parallel and pointing in the opposite direction to $\mathbf{X}_+$, as per their evolving paths in L. The flattening, i.e. rate of change of angle of $\mathbf{X}_-$ and $\mathbf{X}_0$ with respect to $\mathbf{X}_+$, also follows an inverse square law in m. [2017_8]

Following the geometric study some physical aspects are investigated, starting with a dimensional analysis, which shows that $\mathbf{X}_+$ can be regarded as the first derivative of $\mathbf{X}_0$, and the second derivative of $\mathbf{X}_-$, with respect to m. This leads to consideration of $\mathbf{X}_+$ as a constant acceleration vector, $\mathbf{X}_0$ as a velocity vector with a constant acceleration $-\mathbf{X}_+$, and $\mathbf{X}_-$ as a position vector with a constant acceleration $-2\mathbf{X}_+$, twice that of $\mathbf{X}_0$. Consequently, the evolution of $\mathbf{X}_-$ is a point accelerating down $\mathbf{C}_L$, twice that of $\mathbf{X}_0$ down $\mathbf{H}_L$.

Lastly, given the original unity root matrix theory in Ref. [1] was based upon a conservation law and an invariance principle, coupled with the cone and hyperboloid geometry of the eigenvectors, plus a consistent interpretation of the eigenvectors with physical quantities, then Section (14) provided some links to concepts in mathematical physics. Key amongst these concepts is: 1) quantisation, since the work is exclusively in integers; 2) symmetry, conservation laws and local and global transformation invariance; 3) evolutionary and physical scale; and 4), Minkowski geometry.

(16) Conclusions

A relatively simple, integer matrix, with two eigenvectors satisfying the Pythagoras theorem, and a third satisfying a hyperbolic equation, possesses a geometry of sufficient structure to give some interesting geometric properties, e.g. angular evolution and curvature. The eigenvectors also possess a consistent physical interpretation as dynamical quantities such as position, velocity and acceleration with their related calculus. With such properties observed, from what is a relatively basic starting point, it is concluded that the further study of unity root matrices, and associated algebra, may offer a reformulation of some physical phenomena in a simpler, quantised form, without the need for a real or complex vector space under-pinning much of modern, mathematical physics.

- End of Part II –

Acknowledgements
The author would like to thank Micro SciTech Ltd. for funding the work.

References

[1] Unity Root Matrix Theory, Physics in Integers, Richard J. Miller, FastPrint Publishing, 2011, ISBN 978-184426-974-7,

http://www.fast-print.net/bookshop/823/unity-root-matrix-theory-physics-in-integers

This book is broken into six separate papers, each paper is given a specific reference #1 to #6 as follows:

[1]#1 Unity Root Matrix Theory Foundations
[1]#2 see [11], below
[1]#3 Geometric and Physical Aspects
[1]#4 Solving Unity Root Matrix Theory
[1]#5 Unifying Concepts
[1]#6 A Non-unity Eigenvalue

[2] R J Miller, Pythagorean Triples as Eigenvectors and Related Invariants, Micro SciTech Ltd., 2010.

http://www.microscitech.com/pythag_eigenvectors_invariants.pdf

[3] Discrete Integrable Geometry and Physics, edited by Alexander I. Bobenko and Ruedi Seiler, Oxford University Press, 1999, ISBN 0-19-850160-9.

[4] I Niven, S Zuckerman, H L Montgomery, An Introduction to the Theory of Numbers, 5th Edition, John Wiley & Sons, Inc., 1991, ISBN 0-471-54600-3.

[5] R J Miller, A Quark Flavour Model in Integers (Pythagorean Sextuplets as Quarks and Related Invariants).

[6] Sadri Hassani, Foundations of Mathematical Physics, Prentice Hall International, Inc., 1991, ISBN 0-13-327503-5.

[7] Roger Penrose, The Road to Reality, BCA 2004, CN 127408.

[8] H. S. M. Coxeter, Introduction to Geometry, 2nd Edition, John Wiley & Sons, Inc. 1989, 0-471-50458-0.

[IV] [4] Unity Root Matrix Theory, Mathematical and Physical Advances Volume II, Richard J. Miller, New Generation Publishing, 2014, ISBN 978-0-7552-1675-8.

[V] Unity Root Matrix Theory, A Quark Flavour Model, Richard J. Miller, New Generation Publishing, 2016, ISBN 978-1-78719-212-6.

Appendix (A)
Non-unity eigenvalues $\lambda = +C, 0, -C$

This Appendix is a summary of all relevant equations for non-unity eigenvalues, $\lambda = \pm C$, used throughout this paper. The unity equivalent equations for $\lambda = \pm 1$ were first derived in [1] and [2]. There are some additional equations included here that are not used specifically in this paper but are provided for completeness. Obviously, only those equations containing the eigenvalue C are actually modified from [1] and [2], and then, only if using a value $C \neq 1$.

(1.1) $$\mathbf{A} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix},\ P,Q,R \in \mathbb{Z},\ (P,Q,R) \neq (0,0,0).$$

(A1) $$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

(A2) $$-\lambda^3 + \lambda(P^2 + Q^2 - R^2) = 0$$

(2.2.1) $$C^2 = P^2 + Q^2 - R^2,\ C \in \mathbb{Z},\ C > 0$$

(A3) $$\lambda(\lambda - C)(\lambda + C) = 0$$

(A4) $$\lambda_+ = C,\ \lambda_0 = 0,\ \lambda_- = -C$$

(2.1.2) $$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ x,y,z \in \mathbb{Z},\ (x,y,z) \neq (0,0,0)$$

(2.3.1c) $$\mathbf{X}^- = \begin{pmatrix} x & y & -z \end{pmatrix}$$

(2.1.3) $$\mathbf{X}_+ \cdot \mathbf{X}^- = x^2 + y^2 - z^2 = 0$$

(2.1.4) $$\mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix},\ \alpha,\beta,\gamma \in \mathbb{Z},\ (\alpha,\beta,\gamma) \neq (0,0,0)$$

(2.3.1a) $$\mathbf{X}^+ = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$$

(2.1.5) $$\mathbf{X}_- \cdot \mathbf{X}^+ = \alpha^2 + \beta^2 - \gamma^2 = 0$$

(2.1.6) $$\mathbf{X}_0 = \begin{pmatrix} +P \\ -Q \\ +R \end{pmatrix}$$

(2.3.1b) $$\mathbf{X}^0 = \begin{pmatrix} P & -Q & -R \end{pmatrix}$$

(2.2.1) $$\mathbf{X}_0 \cdot \mathbf{X}^0 = P^2 + Q^2 - R^2 = C^2$$

(2.2.2a) $\mathbf{AX}_+ = +C\mathbf{X}_+$

(2.2.2b) $\mathbf{AX}_0 = 0$

(2.2.2c) $\mathbf{AX}_- = -C\mathbf{X}_-$

(2.3.5a) $\mathbf{X}^+\mathbf{A} = +C\mathbf{X}^+$

(2.3.5b) $\mathbf{X}^0\mathbf{A} = 0$

(2.3.5c) $\mathbf{X}^-\mathbf{A} = -C\mathbf{X}^-$.

(2.2.3)

(2.2.3a) $Cx = Ry + Qz$

(2.2.3b) $Cy = -Rx + Pz$

(2.2.3c) $Cz = Qx + Py$

(2.2.3e) $-C\alpha = R\beta - Q\gamma$

(2.2.3f) $-C\beta = -R\alpha - P\gamma$

(2.2.3g) $C\gamma = Q\alpha + P\beta$.

(2.2.4) $\mathbf{X}_+ \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^- = xP - yQ - zR = 0$

(2.2.5) $\mathbf{X}_- \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^+ = \alpha P - \beta Q + \gamma R = 0$

(A5)

(A5a) $y(C^2 - P^2) = (-CR + PQ)x$

(A5b) $z(C^2 - Q^2) = (CP + QR)y$

(A5c) $x(C^2 + R^2) = (CQ + RP)z$

(A5d) $z(C^2 - P^2) = (CQ - RP)x$

(A5e) $x(C^2 - Q^2) = (CR + PQ)y$

(A5f) $y(C^2 + R^2) = (CP - QR)z$

(A5g) $z(-CR + PQ) = y(CQ - RP)$

(A5h) $z(CR + PQ) = x(CP + QR)$

(A5i) $y(CQ + RP) = x(CP - QR)$

(2.2.6)

(2.2.6a) $\alpha x = (C^2 - P^2)$

(2.2.6b) $\beta y = (C^2 - Q^2)$

(2.2.6c) $\gamma z = (C^2 + R^2)$

(2.2.7) $\mathbf{X}^+\mathbf{X}_+ = \mathbf{X}^-\mathbf{X}_- = \alpha x + \beta y + \gamma z = 2C^2$

(A6)

(A6a) $\alpha y = (-CR + PQ)$

(A6b) $\alpha z = (CQ - RP)$

(A6c) $\beta x = (CR + PQ)$
(A6d) $\beta z = (CP + QR)$
(A6e) $\gamma x = (CQ + RP)$
(A6f) $\gamma y = (CP - QR)$

(2.5.1) $k, l \in \mathbb{Z}$, $(k,l) \neq (0,0)$, $\gcd(k,l) \mid C$

(2.5.2a) $x = 2kl$
(2.5.2b) $y = (l^2 - k^2)$
(2.5.2c) $z = (l^2 + k^2)$

(2.5.3) $C = ks - lt$, $s, t \in \mathbb{Z}$

(2.5.4a) $s = s' + ml$
(2.5.4b) $t = t' + mk$

(2.5.5a) $P = -(ks + lt)$
(2.5.5b) $Q = (ls - kt)$
(2.5.5c) $R = -(ls + kt)$

(2.5.6a) $\alpha = -2st$
(2.5.6b) $\beta = (t^2 - s^2)$
(2.5.6c) $\gamma = (t^2 + s^2)$.

With the extension to non-unity eigenvalues comes some modification to the definition of the 'unity roots', i.e. the dynamical variables P, Q, R are no longer roots of unity. For a general eigenvalue C, the dynamical variables P, Q, R now satisfy the following congruence relations

(A7)
(A7a) $P^2 \equiv C^2 \pmod{x}$
(A7b) $Q^2 \equiv C^2 \pmod{y}$
(A7c) $R^2 \equiv -C^2 \pmod{z}$.

When $C = +1$ the dynamical variables P and Q are seen to square to +1, and R squares to -1, hence they are termed unity roots. When $C \neq +1$, the unity root property evidently no longer applies since the dynamical variables no longer square to unity, but C^2 instead, as in (A7). However, from the theory of power residues and primitive roots, see [4], knowledge of the unity roots is sufficient to find any non-unity, quadratic residue. For example, if P' is a unity root such that $P'^2 \equiv +1 \pmod{x}$, then multiplying throughout by C^2 implies $(CP')^2 \equiv +C^2 \pmod{x}$. By defining P as $P \equiv CP' \pmod{x}$ then $P^2 \equiv C^2 \pmod{x}$ and hence P satisfies (A7a). Therefore knowing P' enables P to be determined.

Lastly, note that the special case, illustrated by the almost-trivial solution in Appendix (C), when Q and R are both zero, does not contradict the definitions (A7b) and (A7c). This is because the dynamical variables Q and R, and eigenvalue C, happen to be congruent to zero moduli y and z, both moduli being l^2 in this example, i.e. $Q^2 \equiv (sl)^2 \equiv +C^2 \equiv 0 (\bmod l^2)$ and $R^2 \equiv (-sl)^2 \equiv -C^2 \equiv 0 \ (\bmod l^2)$ since $l \mid C$. So, although Q and R are not true unity roots, neither do they contradict any results. Furthermore $P = C$ for all moduli x, zero or otherwise, so neither does this contradict definition (A7a).

Appendix (B)
Example. Pythagorean Triple (4,3,5)

This Section provides some example data in the case of the simplest, non-trivial Pythagorean triple (4,3,5).

Choose integers k and l subject to (2.5.1), $k,l \in \mathbb{Z}$, $(k,l) \neq (0,0)$, $\gcd(k,l) \mid C$,

(B1) $l = 2$, $k = 1$.

The triple (x, y, z) is then given by the familiar Pythagoras parameterisations (2.5.2), $x = 2kl$, $y = (l^2 - k^2)$ and $z = (l^2 + k^2)$

(B2) $x = 4$, $y = 3$, $z = 5$.

Choose eigenvalue C as unity for simplicity and to compare with [2]

(B3) $C = +1$.

Solve the congruence $C = ks - lt$ (2.5.3) to give a general solution for s, t in terms of an arbitrary, integer parameter m

(B4) $s = 1 + 2m$, $t = m$, $m \in \mathbb{Z}$.

The triple (P, Q, R) can then be obtained from (2.5.5), $P = -(ks + lt)$, $Q = (ls - kt)$ and $R = -(ls + kt)$

(B5) $P = -1 - 4m$
(B6) $Q = 2 + 3m$
(B7) $R = -2 - 5m$.

The divisibility factor triple (α, β, γ) is obtained from (2.5.6), $\alpha = -2st$, $\beta = (t^2 - s^2)$ and $\gamma = (t^2 + s^2)$

(B8) $\alpha = -(4m^2 + 2m)$
(B9) $\beta = -(3m^2 + 4m + 1)$
(B10) $\gamma = (5m^2 + 4m + 1)$.

For the primitive solution $m = 0$, so s and t in (B4) become

(B11) $s = +1$, $t = 0$.

Substituting $m=0$ into (B5) to (B10), the following values for the dynamical variables P,Q,R and scale factors α,β,γ are obtained

(B12) $P=-1$, $Q=2$, $R=-2$

(B13) $\alpha=0$, $\beta=-1$, $\gamma=1$.

With all nine variables $\{x,y,z,P,Q,R,\alpha,\beta,\gamma\}$ assigned, (B2), (B12) and (B13), the standard eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ are, according to definitions (2.1.2), (2.1.6) and (2.1.4) respectively,

(B14) $$\mathbf{X}_+=\begin{pmatrix}4\\3\\5\end{pmatrix},\ \mathbf{X}'_0=\begin{pmatrix}-1\\-2\\-2\end{pmatrix},\ \mathbf{X}'_-=\begin{pmatrix}0\\-1\\-1\end{pmatrix}.$$

The conjugate eigenvectors are, according to (2.3.1),

(B15) $$\mathbf{X}^+=\begin{pmatrix}0 & -1 & +1\end{pmatrix},\ \mathbf{X}'^0=\begin{pmatrix}-1 & -2 & +2\end{pmatrix},\ \mathbf{X}'^-=\begin{pmatrix}4 & 3 & -5\end{pmatrix}.$$

Using (B12) for the dynamical variables P,Q,R, the $\mathbf{A}$ matrix (1.1) is

(B16) $$\mathbf{A}=\begin{pmatrix}0 & -2 & +2\\ +2 & 0 & -1\\ +2 & -1 & 0\end{pmatrix}.$$

Since $z>0$ then $\mathbf{X}_+\in\mathbf{C}_U$ by definition (4.1).

With $z=5$ and $\gamma=1$ then z and γ are of the same sign and, hence, $-\gamma<0$ and $\mathbf{X}_-\in\mathbf{C}_L$ by definition (4.2).

With $R=-2$ then $\mathbf{X}_0\in\mathbf{H}_L$ by definition (4.4).

(B17) Angle Table

This table gives the angles θ_{+-}, θ_{+0} and θ_{0-} (8.7) between the eigenvectors $\mathbf{X}_+$ ($\sim\mathbf{X}_{m+}$) (6.2a), invariant to variations in *m*, $\mathbf{X}_{m0}$ (6.2b) and $\mathbf{X}_{m-}$ (6.2c).

$\mathbf{X}_+=$ (4,3,5)	$\mathbf{X}_{m0}$ (6.2b) ($C=1$)			$\mathbf{X}_{m-}$ (6.2c)				$\theta_{+0}+\theta_{0-}=180$ (8.8)	
m	P	$-Q$	R	α	β	$-\gamma$	θ_{+-}	θ_{+0}	θ_{0-}
0	-1	-2	-2	0	-1	-1	143.130	160.529	19.471
1	-5	-5	-7	-6	-8	-10	168.522	174.232	5.768
2	-9	-8	-12	-20	-21	-29	173.267	176.628	3.372
4	-17	-14	-22	-72	-65	-97	176.320	178.150	1.841
8	-33	-26	-42	-272	-225	-353	178.071	179.035	0.965

16	-65	-50	-82	-1056	-833	-1345	179.012	179.506	0.494
32	-129	-98	-162	-4160	-3201	-5249	179.500	179.750	0.250
64	-257	-194	-322	-16512	-12545	-20737	179.748	179.874	0.126

(B18) Flatness Table

This table gives the flatness parameter ω, as calculated from (8.13), with an approximation of it (estimate $\hat{\omega}_{+-}$) from (8.25), using θ_{+-} from table (B17), converted to radians, to calculate $\overline{\theta}_{+-}$ (8.21). A second approximation is also obtained from (8.32) using θ_{0-}, also from table (B17), and also converted to radians.

evolution parameter	flatness parameter ω (8.13)	ω approx-imated from $\overline{\theta}_{+-}$ (8.25)	% error	$\hat{\omega}_{0-}$ approx-imated from θ_{0-} (8.32)	% error
m	$\omega = C/R$, ($C=1$)	$\hat{\omega}_{+-} = \frac{-\overline{\theta}_{+-}}{\sqrt{2}}$	$100\left\lvert\frac{\hat{\omega}_{+-}-\omega}{\omega}\right\rvert$	$\hat{\omega}_{0-} \approx -\sqrt{2}\theta_{0-}$	$100\left\lvert\frac{\hat{\omega}_{0-}-\omega}{\omega}\right\rvert$
0	-0.5000	-0.4550	9.00	-0.4806	3.90
1	-0.1429	-0.1417	0.84	-0.1424	0.34
2	-0.0833	-0.0831	0.29	-0.0832	0.12
4	-0.0455	-0.0454	0.086	-0.0454	0.034
8	-0.023810	-0.023804	0.024	-0.023807	0.0094
16	-0.012195	-0.012194	0.0062	-0.012195	0.0025
32	-0.00617284	-0.00617274	0.0016	-0.006173	0.00064
64	-0.00310559	-0.00310558	0.0004	-0.003106	0.00016

(B19) Curvature Table

This table gives the curvature κ, as calculated from (9.2) using θ_{+-} (8.7b), with an approximation of it (estimate $\hat{\kappa}$) from (9.5).

evolution parameter	θ_{+-} (deg) (8.7b)	angle θ_{+-} (rad)	curvature κ (9.2)	$\hat{\kappa}$ approximated (9.5), for $C=1$, $z=5$	% error
m			$\theta_{(m)+-} - \theta_{(m-1)+-}$	$\hat{\kappa} \approx \left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m^2}$	$100\left\lvert\frac{\hat{\kappa}-\kappa}{\kappa}\right\rvert$
0	143.130	2.498092	-	-	-
1	168.522	2.941258	0.443166	0.2828427	36.0
2	173.267	3.024081	0.0828232	0.0707107	15.0
4	176.320	3.077365	0.0188422	0.0176777	6.2
8	178.071	3.107929	0.00454656	0.00441942	2.8
16	179.012	3.124347	0.00111968	0.00110485	1.3
32	179.500	3.132863	0.00027800	0.00027621	0.64
64	179.748	3.137201	0.00006927	0.00006905	0.32

(B20) Mixed Angle Table

This table gives the mixed angles θ_+^+, θ_0^0 and θ_-^- (10.1), between the standard and conjugate, evolving eigenvectors (6.2) and (6.3). Note that $\theta_+^+ = \theta_-^-$ by (10.1a) and (10.1b).

$\mathbf{X}_+ =$ (4,3,5)	$\mathbf{X}^{m0}$ (6.3b) ($C=1$)			$\mathbf{X}^{m+}$ (6.3c)			θ_+^+, θ_-^-	θ_0^0
m	P	$-Q$	$-R$	α	β	γ	(10.1a,b)	(10.1c)
0	-1	-2	2	0	-1	1	78.463	83.261
1	-5	-5	7	-6	-8	10	88.854	89.421
2	-9	-8	12	-20	-21	29	89.605	89.802
4	-17	-14	22	-72	-65	97	89.882	89.941
8	-33	-26	42	-272	-225	353	89.968	89.984
16	-65	-50	82	-1056	-833	1345	89.991	89.996
32	-129	-98	162	-4160	-3201	5249	89.998	89.9989
64	-257	-194	322	-16512	-12545	20737	89.999	89.9997

(B21) Mixed Angle Flatness Table

This table gives the flatness parameter ω, as calculated from (8.13), with an approximation of it (estimate $\hat{\omega}_+^+$) from (10.9), using θ_+^+ (10.1a) to calculate $\bar{\theta}_+^+$ (10.5) in radians.

evolution parameter	θ_+^+ (deg) (10.1a)	angle $\bar{\theta}_+^+$ (rad) (10.5)	flatness parameter ω (8.13)	$\hat{\omega}_+^+$ approx-imated from $\bar{\theta}_+^+$ (10.9)	% error
m		$\bar{\theta}_+^+ = (\pi/2 - \theta_+^+)$	$\omega = C/R$, ($C=1$)	$\hat{\omega}_+^+ = -\sqrt{\bar{\theta}_+^+}$	$100\left\lvert\frac{\hat{\omega}_+^+ - \omega}{\omega}\right\rvert$
0	78.463	0.201358	-0.5000	-0.448729	10.0
1	88.854	0.020001	-0.1429	-0.141426	1.0
2	89.605	0.006897	-0.0833	-0.083046	0.35
4	89.882	0.002062	-0.0455	-0.045408	0.17
8	89.968	0.000567	-0.023810	-0.023803	0.028
16	89.991	0.000149	-0.012195	-0.012194	0.0075
32	89.998	0.000038	-0.006173	-0.006173	0.00020
64	89.999	0.000001	-0.003106	-0.003106	0.000074

(B22) Mixed Angle Curvature Table

This table gives the mixed angle curvature κ_+^+ as calculated from the finite difference $\theta_{(m)+}^+ - \theta_{(m-1)+}^+$, using (10.1a) for θ_+^+, and with an approximation of it (estimate $\hat{\kappa}_+^+$) from (10.15).

evolution parameter	θ_+^+ (deg) (10.1a)	angle θ_+^+ (rad) (10.1a)	curvature κ_+^+	$\hat{\kappa}_+^+$ approximated (10.15), for $C=1$, $z=5$	% error

m			$\theta^{+}_{(m)+} - \theta^{+}_{(m-1)+}$	$\kappa^{+}_{+} = \left(\frac{2C^2}{z^2}\right)\frac{1}{m^3}$	$100\left\lvert\frac{\hat{\kappa} - \kappa}{\kappa}\right\rvert$
0	78.463	1.369438	-	-	-
1	88.854	1.550795	0.181357	0.08	56.0
2	89.605	1.563900	0.013105	0.01	24.0
4	89.882	1.568734	0.001386	0.0125	14.0
8	89.968	1.570230	0.000163	0.000156	4.3
16	89.991	1.570648	0.00001994	0.00001953	2.0
32	89.998	1.570758	0.00000247	0.00000244	1.0
64	89.999	1.570787	0.00000031	0.00000031	0.48

Appendix (C)
The 'almost trivial' solution $(0,1,1)$.

Given the most trivial value $(0,0,0)$, for all three triples (x, y, z), (P, Q, R), (α, β, γ), is excluded as invalid, see Section (5), then the next simplest solution is considered to be the triple $(x, y, z) = (0,1,1)$. Of course, with extensions, see Appendix (D) in [2], any of the four sign combinations are possible, but only the all-positive triple will be used here for simplicity. Note too that $(x, y, z) = (1,0,1)$ is also possible but considered one and the same as $(1,0,1)$, at least in this example, given the symmetry between x and y in Pythagoras.

As $(0,1,1)$ is not generally considered a true Pythagorean triple, it is not given the status of the more familiar, smallest primitive triple $(4,3,5)$, which is examined in Appendix (B). Nevertheless, the triple $(0,1,1)$ does satisfy definition (1.0) and is therefore not to be dismissed lightly.

Instead of using exactly $(0,1,1)$, the more general variant $(0,l,l)$, $l \in \mathbb{Z}$, $l \neq 0$, will be used in this example. With this choice, the integer k, in (2.5.1), is zero and integer l remains unspecified, but both subject to the following conditions, for general eigenvalue C,

(C1) $$k = 0\,,\ l \mid C\,,\ k, l \in \mathbb{Z},\ l \neq 0\,.$$

The condition $l \mid C$ in (C1) ensures integer, not rational, solutions. Obvious choices are $l = \pm 1$ or $l = \pm C$.

The triple (x, y, z) is then given by the familiar Pythagoras parameterisations (2.5.2) $x = 2kl$, $y = (l^2 - k^2)$ and $z = (l^2 + k^2)$

(C2) $$x = 0\,,\ y = l^2\,,\ z = l^2\,.$$

For full generality the eigenvalue C will be left unspecified and not set to unity as in Appendix (B) or [2].

With $k = 0$, the congruence $C = ks - lt$ (2.5.3) becomes simply $t = -C/l$. Integer s is arbitrary and will also be left as such for now, acting as a free parameter.
Formalising this, the general solution to $C = ks - lt$ (2.5.3), arbitrary $s \in \mathbb{Z}$, is

(C3) $$t = -C/l\,,\ l \mid C\,.$$

The triple (P, Q, R) is obtained from (2.5.5), $P = -(ks + lt)$, $Q = (ls - kt)$ and $R = -(ls + kt)$, parameterised by s,

(C4)
(C4a) $$P = C$$

(C4b) $Q = sl$
(C4c) $R = -sl$.

The divisibility factor triple (α, β, γ) is obtained from (2.5.6), $\alpha = -2st$, $\beta = (t^2 - s^2)$ and $\gamma = (t^2 + s^2)$, also parameterised by s, as follows

(C5)

(C5a) $$\alpha = 2s\frac{C}{l}$$

(C5b) $$\beta = \frac{C^2}{l^2} - s^2$$

(C5c) $$\gamma = \frac{C^2}{l^2} + s^2.$$

Using these results, the full suite of standard eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ is, in accordance with (2.1.2), (2.1.6) and (2.1.4) respectively,

(C6) $$\mathbf{X}_+ = \begin{pmatrix} 0 \\ l^2 \\ l^2 \end{pmatrix}, \ \mathbf{X}_0 = \begin{pmatrix} +C \\ -sl \\ -sl \end{pmatrix}, \ \mathbf{X}_- = \frac{1}{l^2}\begin{pmatrix} 2slC \\ C^2 - s^2 l^2 \\ -(C^2 + s^2 l^2) \end{pmatrix}.$$

The $\mathbf{A}$ matrix (1.1) is

(C7) $$\mathbf{A} = \begin{pmatrix} 0 & -sl & +sl \\ +sl & 0 & +C \\ +sl & +C & 0 \end{pmatrix}.$$

By comparison of the evolution equations (6.2) with eigenvectors (C6), it can be inferred that the evolution parameter m is related to integer parameter s by

(C8) $s = ml$.

However, do not substitute for s, using ml (C8), as it will not give all solutions, only those where s is a multiple of l. Consequently, (C6) is left as is and the 'evolution parameter' is effectively s here. Evidently s and m can be made equal by setting $l = 1$, which is done for the example data tabulated further below, tables (C13) to (C16).

For the primitive solution, $s = 0$, it is seen that the two dynamical variables Q (C4b) and R (C4c) are both zero. This is the exceptional case where they are most definitely not unity roots but, that aside, are still perfectly valid and consistent values as regards satisfying all related equations, notably the conservation equation (2.2.1) and divisibility criteria (2.2.6).

The primitive case $s=0$ also reduces $\mathbf{A}$ to having only two, non-zero elements, both C but, nevertheless, it still gives a consistent set of dynamical equations (2.2.2).

The conjugate eigenvectors are, according to definitions (2.3.1),

(C9) $\mathbf{X}^{+} = \frac{1}{l^2}\begin{pmatrix} 2slC & C^2 - s^2l^2 & C^2 + s^2l^2 \end{pmatrix}$

(C10) $\mathbf{X}^{0} = \begin{pmatrix} C & -sl & +sl \end{pmatrix}$

(C11) $\mathbf{X}^{-} = \begin{pmatrix} 0 & l^2 & -l^2 \end{pmatrix}$.

The most interesting aspect of this example is that the eigenvectors for $s=0$ are orthogonal and thus form a right-handed triad. Although, without scaling, they are not unit vectors and, hence, the basis is not orthonormal

(C12) $\mathbf{X}_{+} = \begin{pmatrix} 0 \\ l^2 \\ l^2 \end{pmatrix}$, $\mathbf{X}_{0} = \begin{pmatrix} C \\ -sl \\ -sl \end{pmatrix}$, $\mathbf{X}_{-} = \frac{1}{l^2}\begin{pmatrix} 0 \\ C^2 \\ -C^2 \end{pmatrix}$, $s=0$.

The magnitudes of $\mathbf{X}_{+}$, $\mathbf{X}_{0}$ and $\mathbf{X}_{-}$ in (C12) are $\sqrt{2}l^2$, C and $\sqrt{2}(C/l)^2$, but the eigenvectors are never normalised within the context of unity root matrix theory since this takes it out of the integer domain into the reals, and tantamount to heresy within the context of this work.

(C13) Angle Table

The following tables are for the above, almost-trivial solution - See (B17) for a description of the tables.

$\mathbf{X}_{+} =$ (0,1,1) $l=1$ (C12)	$\mathbf{X}_{m0}$ (6.2b) ($C=1$)			$\mathbf{X}_{m-}$ (6.2c)				$\theta_{+0} + \theta_{0-} = 180$	
m	P	$-Q$	R	α	β	$-\gamma$	θ_{+-}	θ_{+0}	θ_{0-}
0	1	0	0	0	1	-1	90.0	90.0	90.0
1	1	-1	-1	2	0	-2	120.0	144.736	35.264
2	1	-2	-2	4	-3	-5	143.130	160.529	19.471
4	1	-4	-4	8	-15	-17	160.25	169.975	10.025
8	1	-8	-8	16	-63	-65	169.937	174.949	5.051
16	1	-16	-16	32	-255	-257	174.944	177.470	2.530
32	1	-32	-32	64	-1023	-1025	177.469	178.734	1.266
64	1	-64	-64	128	-4095	-4097	178.734	179.367	0.633

(C14) Flatness Table

See (B18) for description.

evolution parameter $m = s/l$, $l, s = 1$ (C8)	flatness parameter ω (8.13)	ω approx-imated from $\overline{\theta}_{+-}$ (8.25)	% error	$\hat{\omega}_{0-}$ approx-imated from θ_{0-} (8.32)	% error
m	$\omega = C/R$ ($C = 1$)	$\hat{\omega}_{+-} \approx \frac{-\overline{\theta}_{+-}}{\sqrt{2}}$	$100\left\lvert\frac{\hat{\omega}_{+-} - \omega}{\omega}\right\rvert$	$\hat{\omega}_{0-} \approx -\sqrt{2}\theta_{0-}$	$100\left\lvert\frac{\hat{\omega}_{0-} - \omega}{\omega}\right\rvert$
0	∞ ($R = 0$)	-1.110	-	-2.221	-
1	-1.0	-0.740	26.0	-0.870	13
2	-0.5	-0.455	9.0	-0.481	3.9
4	-0.25	-0.244	2.5	-0.247	1.0
8	-0.125	-0.12419	0.64	-0.12468	0.26
16	-0.0625	-0.06240	0.16	-0.06246	0.065
32	-0.03215	-0.031237	0.041	-0.031245	0.016
64	-0.015625	-0.0156234	0.01	-0.0156244	0.0041

(C15) Curvature Table

See (B19) for description.

evolution parameter $m = s/l$, $l, s = 1$ (C8)	θ_{+-} (deg) (8.7b)	angle θ_{+-} (rad)	curvature κ (9.2)	$\hat{\kappa}$ approximated (9.5), for $C = 1$, $z = 1$	% error
m			$\theta_{(m)+-} - \theta_{(m-1)+-}$	$\kappa \approx \left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m^2}$	$100\left\lvert\frac{\hat{\kappa} - \kappa}{\kappa}\right\rvert$
0	90.0	1.5708	-	-	-
1	120.0	2.0940	0.5232	1.4142	170
2	143.130	2.4981	0.4041	0.3535	12
4	160.25	2.7969	0.1063	0.0884	17
8	169.937	2.9660	0.0247	0.0221	11
16	174.944	3.053347	0.005862	0.005524	5.8
32	177.469	3.097416	0.001424	0.001381	3.0
64	178.734	3.119498	0.000351	0.000345	1.5

(C16) Mixed Angle Table

See (B20) for description.

$\mathbf{X}_+ =$ (0,1,1), $l=1$	$\mathbf{X}^{m0}$ (6.3b) ($C=1$)			$\mathbf{X}^{m+}$ (6.3c)			θ_+^+, θ_-^-	θ_0^0
m	P	$-Q$	-R	α	β	γ	(10.1a,b)	(10.1c)
0	1	0	0	0	1	1	0.0	0.0
1	1	-1	1	2	0	2	60.0	70.529
2	1	-2	2	4	-3	5	78.463	83.621
4	1	-4	4	8	-15	17	86.628	88.264
8	1	-8	8	16	-63	65	89.119	89.556
16	1	-16	16	32	-255	257	89.777	89.888
32	1	-32	32	64	-1023	1025	89.944	89.972
64	1	-64	64	128	-4095	4097	89.986	89.993

The remaining two, equivalent tables to (B21) and (B22) in Appendix (B) have not been added to this Appendix (C) as unnecessary. They show the same trends as per (B21) and (B22).

Appendix (D)
2017 Revision

[2017_1] The connection to angular momentum and spin is firmly established in [IV] and [V]. In the former, the dynamical variables P, Q, R are related to quaternions and hence too rotations, whilst in [5] they relate to particle spin.

[2017_2] X+ is invariant under time-domain evolution, parameter m (or t). But latter URMT introduces dual eigenvector evolution, which is also as the frequency-domain evolution, and in this case it is the vector X- that is invariant, with X+ and X0 evolving in terms of a frequency parameter.

[2017_3] QPI/SPI

[2017_4] Physical advances have moved on considerably.

[2017_5] Trace-free Generators, Books 4 and 5.

[2017_6] Spin was borne out, books 4 and 5.

[2017_7] 4D and 5D extended in Book 2 and showed Geometric Compactification. STR 4D and 5D came in Book 3.

[2017_8] This is basically another form of compactification.