Unity Root Matrix Theory
Comprehensive Index

Richard J. Miller
Email: richard@microscitech.com,
www.urmt.org

This document provides a comprehensive index for all five books currently published on the subject of Unity Root Matrix Theory, none of which have such an index at the end.

The five published books are as follows:

[1] Physics in Integers
[2] Higher Dimensional Extensions
[3] Mathematical and Physical Advances Volume I
[4] Mathematical and Physical Advances Volume II
[5] A Quark Flavour Model

In addition to an index, this document also contains a glossary of terms and definitions following the index. This glossary was first added as Appendix (I) in book [2], and has evolved in each book thereafter, with the latest and most up-to-date in book [5], as reproduced herein. Note that it is not, by design, a complete URMT Glossary, but merely covers terms either uniquely defined within URMT or those in wider use, but with a specific meaning to URMT.

Key

[B] nnn = book B (B=1-5), page nnn

Glossary (I*nn*) = Glossary entry *nn*

Multiple entries are separated by commas.

For example

[1] 103: Book I, page 103

[1] 126, [1] 200, [2] 275: Book 1 page 126, Book 1 page 200, book 2 page 275

C

D

E

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X

Z

Glossary of Terms and Definitions
Appendix (I) books [2] to [5]

This is a subset of the full URMT terminology covering terms either uniquely defined within URMT or those in wider use, but with a specific meaning to URMT.

Important. The ordering of entries is alphabetic, not numeric, and is done to achieve compatibility between all five published books. To aid in its use, both an alphabetic and numeric index is given beforehand on the next two pages.

Alphabetic Index

Numeric Index

(I22) **A matrices**

The definition of **A matrices**, as first given in Volume I [3], was exclusively defined in terms of matrices formed from the method of arbitrary embedding AVE I, which was new to URMT at the time. This AVE I method produces an $(n+1)\times(n+1)$, $n\geq 1$, **A** matrix defined in terms of an arbitrary, n-element column vector **X**, its row-vector transpose $\mathbf{X}^{\mathrm{T}}$ and an **annihilator** matrix Δ (I23) of **X**; see Section [4],1 for a full review of this method.

As of the last publication [4], the term **A** matrix was extended to all URMT matrices, specifically the founding 'unity root matrix' **A** (~$\mathbf{A_0}$) and its two partners, the plus and minus matrices $\mathbf{A}_+$ and $\mathbf{A}_-$. These three matrices formed the complete set of **A** matrices under study in URMT until this latest publication. The matrix $\mathbf{A_0}$ is the starting point, and the matrices $\mathbf{A}_+$ and $\mathbf{A}_-$ can be considered subsidiary, generated from the eigenvectors of $\mathbf{A_0}$. This founding set of three is now known as the **Base** set $\{\mathbf{A}_+,\mathbf{A}_0,\mathbf{A}_-\}$ (I41), to differentiate it from two, new additional sets, each comprising three **A** matrices, termed the **Difference** set $\{\mathbf{A}_{D+},\mathbf{A}_{D0},\mathbf{A}_{D-}\}$ (I42) and the **Sum** set $\{\mathbf{A}_{S+},\mathbf{A}_{S0},\mathbf{A}_{S-}\}$ (I43). Each set of three can be related to an axis in three-dimensional space, i.e. the *x*, *y* and *z*-axis, three matrices per axis. The two matrices subscripted '+ and '-' are equivalent to raising and lowering operators in quantum mechanics, and the zero subscripted matrices are unity root matrices, e.g. $\mathbf{A_0}$, whose eigenvectors also comes in three sets of plus, minus and zero forms, i.e. $\{\mathbf{X}_+,\mathbf{X}_0,\mathbf{X}_-\}$, $\{\mathbf{X}_{D+},\mathbf{X}_{D0},\mathbf{X}_{D-}\}$ and $\{\mathbf{X}_{S+},\mathbf{X}_{S0},\mathbf{X}_{S-}\}$. Each zero matrix $\{\mathbf{A}_0,\mathbf{A}_{D0}\mathbf{A}_{S0}\}$ can be thought of as a generator of rotation about an axis $\{x,y,z\}$, as per quantum mechanical angular momentum or spin.

Returning now to the legacy, founding unity root matrix **A** (~$\mathbf{A_0}$). URMT starts with the single, 3x3 unity root matrix **A** (A1a) whose elements, known as the dynamical variables, are all unity roots or power residues (A2). This matrix, in its most general form, has links to both Fermat's Last Theorem (via the invariant eigenvector) and The Riemann Hypothesis (via its eigenvalues). However, URMT quickly steered away from its number-theoretic origins to physics, most of which comes when the unity root matrix theory is simplified under **Pythagoras conditions** (I13) or, more recently, **Skew conditions** (I33).

This and the last two publications, [3] and [4], study URMT and its **A** matrix eigenvector solution exclusively under Pythagoras or Skew conditions, and it can be assumed that one of these two conditions always applies unless stated otherwise. Note that they are mutually exclusive.

URM3 encapsulates most of the general properties of URMT, and the following properties are illustrated by reference to URM3 detailed in Appendix (A), but note that Appendix (A) only specifically deals with $\mathbf{A_0}$. More detailed information on all three types of matrices $\{\mathbf{A}_+,\mathbf{A}_0,\mathbf{A}_-\}$ can be found in the main sections of this book - references given shortly.

The elements of $\mathbf{A_0}$ are known as dynamical variables, e.g. P,Q,R (A1a) URM3.

The elements of $\mathbf{A}_+$ are known as coordinates, e.g. x, y, z (A3) URM3

The elements of $\mathbf{A}_-$ are known as scale factors (I2), e.g. α, β, γ (A13) URM3

Each of the three matrices, $\mathbf{A_0}$, $\mathbf{A}_+$ and $\mathbf{A}_-$, annihilate the eigenvectors $\mathbf{X_0}$, $\mathbf{X}_+$ and $\mathbf{X}_-$ (A33) of $\mathbf{A_0}$, i.e.

$$\mathbf{A}_+\mathbf{X}_+ = 0,\ \mathbf{A}_0\mathbf{X}_0 = 0,\ \mathbf{A}_-\mathbf{X}_- = 0$$

and thus each matrix has at least one, zero eigenvector.

General, *n*-dimensional **A** matrices, with the above properties, can currently be constructed by the method of lifting (I6), or AVE I and II, see [4]. Lifting of the URM3 solution is used in this latest publication to produce the URM6, six-quark solution, Section (13). Conversely, AVE methods have not been required herein.

All three **A** matrices (plus, minus and zero) have a zero trace and, as a consequence, their eigenvalues always sum to zero. In fact, when using the exterior product formulation [4],5 it is shown that the matrix $\mathbf{A_0}$ has two non-zero eigenvectors, $\lambda = \pm C$, with all others zero. The other two matrices, $\mathbf{A}_+$ and $\mathbf{A}_-$, have all *n* eigenvalues zero.

The true, founding, unity root matrix **A** (or $\mathbf{A_0}$, $\mathbf{A_{30}}$) only exists for the integer formulation of URM3 (and not higher order incarnations URM4 and beyond) when the invariant eigenvalue C (A4) is unity. For all other integer values, $C > 1$, the unity roots are nth order power residues, and quadratic residues when under Pythagoras conditions (I13). When **A** is under Pythagoras conditions it is also denoted by symbol $\mathbf{A_0}$ or $\mathbf{A_{30}}$. The subscript '0' denotes **A** when it is under Pythagoras conditions, and the additional subscript '3' denotes it is a 3x3, URM3 matrix, where each subscript may or may not be employed depending on the context.

All **A** matrices have a sign symmetry about the lead diagonal, which is either Pythagorean (I23c) or skew-symmetric (I23b). There is no explicit symmetric **A** matrix as it has currently has no physical application in URMT.

See also **Annihilators** (I23), the **Dynamical equations** (I21), **Lifting** (I6) and **Variational methods** (I20).

(I1) **Alignment** - see **Flattening** (I5).

(I23) An **Annihilator** is defined as any $n \times n$ matrix, general symbol Δ, that is symmetric, barring the sign of its elements, with an all-zero lead diagonal that, when operating on an invariant eigenvector **X** of **A**, eigenvalue λ, reduces it to zero, i.e.

(I23a)

$$\text{if } \mathbf{AX} = \lambda\mathbf{X} \text{ then } \mathbf{\Delta X} = 0,\ \left|\Delta_{ij}\right| = \left|\Delta_{ji}\right|, i, j = 1..n,\ \Delta_{ij} = 0,\ i = j.$$

To all intents and purposes, Δ is either a plus or minus matrix, i.e. $\mathbf{A}_+$ or $\mathbf{A}_-$ when using the **Base** set (I41).

Caution. This definition (I23a) of an annihilator matrix is that strictly used in [1] and [2], but has been broadened slightly since the method of arbitrary vector embedding (AVE I) was introduced in Volume I [3],1 and widened to AVE II in [4]. A looser definition is now used to specify any matrix that annihilates an arbitrary vector **X**, which is not necessarily an eigenvector of any **A** matrix. The text that follows here in this glossary entry is still the strict definition, but the definition and context in which Δ is used throughout should be sufficient to determine its strict adherence or otherwise to this description.

From the above definition (I23a), **X** is a zero eigenvector (I18) of the annihilator matrix $\mathbf{\Delta}$ for eigenvalue zero. In linear algebra **X** is said to occupy the kernel (or null) space of Δ; see [5].

If $\lambda = 0$ then **A** is also an annihilator of $\mathbf{X}$ by definition.

The key property of an annihilator matrix is that, when added to matrix **A**, it leaves the eigenvector **X** and eigenvalue λ invariant, i.e.

$$\mathbf{AX} = (\mathbf{A} + \mathbf{\Delta})\mathbf{X} = \lambda\mathbf{X}$$

Because of this invariant property, Δ can be used to vary **A** whilst keeping **X** and λ constant, and is thus also known as a variational matrix in URMT and described in full in [1]#5,2 and [2],8.

Whilst it leaves **X** and λ invariant, the other eigenvectors and eigenvalues of **A** generally vary (evolve) according to the parameterisation of Δ, and it is the study of their evolution that is of interest in URMT.

All **A** matrices, unity root matrices and annihilators come in two sign forms, Pythagoras and Skew:

(I23b) $\mathbf{\Delta}^S$ Skew sign structure,

$-\mathbf{\Delta}^S = \left(\mathbf{\Delta}^S\right)^T$, skew or anti-symmetric

Example

$$\mathbf{\Delta}^S = \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ +y & -x & 0 \end{pmatrix}, \; sg(\mathbf{\Delta}^S) = \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{pmatrix}.$$

(I23c) $\mathbf{\Delta}^{\mathbf{p}}$ Pythagoras sign form

The Pythagoras sign-form is defined in terms of a skew symmetric matrix operated upon by a Pythagoras **T** operator (I28a), equivalent to the Minkowski metric in Special Relativity [9].

$\mathbf{\Delta}^P = \mathbf{T}\mathbf{\Delta}^S$, for Pythagoras **T** operator (I28a).

Example

$$\mathbf{\Delta}^P = \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ -y & +x & 0 \end{pmatrix}, sg(\mathbf{\Delta}^P) = \begin{pmatrix} 0 & + & - \\ - & 0 & + \\ - & + & 0 \end{pmatrix}.$$

Although not of particular significance, the Pythagoras form can be split into the following sum of a symmetric and skew (anti-symmetric) matrix:

$$\mathbf{\Delta}^P = \begin{pmatrix} 0 & 0 & -y \\ 0 & 0 & +x \\ -y & +x & 0 \end{pmatrix} + \begin{pmatrix} 0 & +z & 0 \\ -z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(I41) The **Base set** of eigenvectors and **A** matrices (I22) comprises the eigenvectors $\{\mathbf{X}_+, \mathbf{X}_0, \mathbf{X}_-\}$ and matrices $\{\mathbf{A}_+, \mathbf{A}_0, \mathbf{A}_-\}$ respectively, defined in full below, in terms of the coordinates (x, y, z) (A3), dynamical variables (P, Q, R) (A18), and scale factors (I15) (α, β, γ)

$$\mathbf{A}_+ = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ -y & x & 0 \end{pmatrix}, \mathbf{A}_0 = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix} \text{(A19)},$$

$$\mathbf{A}_- = \begin{pmatrix} 0 & -\gamma & -\beta \\ \gamma & 0 & \alpha \\ -\beta & \alpha & 0 \end{pmatrix}$$

$$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \mathbf{X}_0 = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix}, \mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}, \text{(A33)}$$

The eigenvectors and matrices are the founding matrices of URMT under Pythagoras Conditions (I13). The base set of **A** matrices forms one of three sets of **A** matrices, and represents the *x*-axis components of a three-axis scheme, whereby the other two sets, i.e. the **Difference** (I42) and **Sum** (43) sets represent the *y* and *z*-axis components respectively.

(I46) A Bilinear form '**B**' in URMT is simply a matrix formed from the outer product of two vectors (L6d), or linear sum of such products, i.e. if $\mathbf{X}$ and $\mathbf{Y}$ are column vectors, with $\mathbf{Y}^T$ the row-vector, transpose of **Y**, then the bilinear form $\mathbf{B}$ is simply the outer product

(I46a) $\mathbf{B} = \mathbf{X}\mathbf{Y}^T$.

This satisfies the more formal, conventional definition of a bilinear form as a function that takes two vector arguments and returns a scalar. For example, if **U** and **V** represent two arbitrary vectors with the same dimensions as **X** and **Y** then

(I46b) $\mathbf{B}(\mathbf{U},\mathbf{V}) = \mathbf{U}^T\left(\mathbf{X}\mathbf{Y}^T\right)\mathbf{V} = \left(\mathbf{U}^T\mathbf{X}\right)\left(\mathbf{Y}^T\mathbf{V}\right) = \text{scalar},$

since both $\mathbf{U}^T\mathbf{X}$ and $\mathbf{Y}^T\mathbf{V}$ are inner products (L6b), resulting in a scalar, such that the function (bilinear form) **B** returns a scalar. There are additional linearity criteria on **B** as per a vector space, for which the reader is referred to the literature, e.g. [5].

(I32) **Compactification** is the process whereby all excess dimensions, i.e. those higher than three, appear to shrink relative to URM3 as their evolutionary parameter grows. In fact, URM3 also **flattens** (I5) as its evolutionary parameter t_3 grows. This geometric feature is a URMT property of lifted solutions (I6). Note that with this book comes the dual, frequency-domain evolution, which can also give the same behaviour as the evolutionary, frequency parameters grow larger in magnitude.

Each dimension, three and higher, possesses its own unique evolutionary time, i.e. t_3 for URMT3, t_4 for URM4 etc., and, just like URM3, converges (flattens) to align with the URM3 cone $\mathbf{C}_U$ (I27) as evolution progresses. The higher dimensions also appear to shrink relative to URM3 as their evolutionary times, t_4, t_5 etc. grow - the entire *n*-dimensional solution exhibiting the geometric property of compactification. See [2],14 for more information.

(I27) The **Cone** comprises the eigenvector solution set of all plus (I11) and minus eigenvectors (I9) when under Pythagoras conditions (I13). Using URM3 as an example, under such Pythagoras conditions the eigenvectors are Pythagorean triples, specified as a two-parameter family, and form a discrete cone in 3D Euclidean space. In URMn this extends to an *n*-dimensional 'cone'. By URMT convention, the set of all points representing the plus eigenvector solution is termed the upper cone, symbol $\mathbf{C}_U$, whilst the lower cone, symbol $\mathbf{C}_L$, comprises the set of points representing the minus eigenvector. The union of the two sets is referred to as the cone (cones), symbol $\mathbf{C}$, i.e.

$$\mathbf{C} = \mathbf{C}_U \cup \mathbf{C}_L .$$

The upper and lower cones tend to be thought of as mirror images of each other since they are inverted with respect to each other, with their conceptual tips meeting at the origin, albeit the tip $\mathbf{X}_+ = (0,0,0)^T$ and $\mathbf{X}_- = (0,0,0)^T$ is not defined in URMT, see (I34), hence it is conceptual only. Likewise, the cones are not a true mirror image of each other but they do still point in opposing directions by their stricter definition given in [1]#3,4.

See [1]#3,4 for a full definition.

See also **Hyperboloid** (I26) and **Lattice** (I25).

(I42) The **Difference set** of eigenvectors $\{\mathbf{X}_{D+}, \mathbf{X}_{D0}, \mathbf{X}_{D-}\}$ and **A** matrices (I22) $\{\mathbf{A}_{D+}, \mathbf{A}_{D0}, \mathbf{A}_{D-}\}$ is the *y*-axis equivalent of the *x*-axis **Base** set (I41) and *z*-axis **Sum** set (I43).

(I2) **Divisibility factors**, also known as **scale factors** (I15), are the last three elements of the URM n eigenvector $\mathbf{X}^{n+}$ (or $\mathbf{X}_{n-}$), denoted by α_n, β_n and γ_n, $n \geq 3$, see (A15) for URM3, [2],3.56 for URM4 and [2],C16 for UMR n.

(I3) The **Dyadic product** is synonymous with the outer product (L6d) of two vectors. In the context of URMT, the dyadic product of two vectors, $\mathbf{X}_i$ and $\mathbf{X}^j$, $i, j = 1 \ldots n$, gives a square matrix, $\mathbf{M}_{ij} = \mathbf{X}_i \mathbf{X}^j$, of size $n \times n$. The term 'dyadic product' is quite old and is replaced by 'outer product' in modern texts. Nevertheless, the name appears in the earlier URMT literature, e.g. [1], albeit it has been replaced in [2] and herein with the term 'outer product'.

(I19) The **Dynamical Conservation Equation (DCE)** is the non-singular condition on the matrix **A** possessing solutions, i.e.

(I19a) $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$, eigenvalue λ, $\lambda = \sqrt{T_n} C$,

for $\sqrt{T_n}$ see further below,

and is known as the characteristic equation (or characteristic polynomial) in matrix algebra, [5]. This equation can be considered as the founding equation upon which URMT can be derived, albeit this is not the only approach.

The DCE is actually treated as an energy conservation equation (hence its name), and is usually expressed in terms of a kinetic energy term *K* and potential energy term V (per unit mass or inertia). In addition, from [4] onward, the DCE is given for both the Pythagoras (I13) and Skew (I33) conditions, which are differentiated by the usage of the **Parity operator** T_n (I39), (new to [4]), briefly summarised as follows:

$T_n = -1$, $\sqrt{T_n} = i$, Skew conditions
$T_n = +1$, $\sqrt{T_n} = +1$, Pythagoras conditions.
$T_n^2 = +1$.

The key URM n incarnations of the DCE as a characteristic equation (I19a) are

(I19b) $0 = -\lambda^3 + K\lambda + \sqrt{T_n} VC$, URM3, see also (A7)-(A9)

(I19c) $0 = -\lambda^4 + K\lambda^2 + VT_nC^2$, URM4 (lifts URM3)

(I19d) $0 = \lambda\left(-\lambda^4 + K\lambda^2 + VT_nC^2\right)$, URM5 (lifts URM4).

The URM6, six-quark solution has a more restrictive form of characteristic equation, with the eigenvector solution termed an **Invariant Zero Potential** (IZP) solution (I35) in URMT. This IZP is also applicable to URM3 onward, and has a general form of the DCE given by

(I19e) $0 = \lambda^{n-2}\left(T_nC^2 - \lambda^2\right),\ n \geq 3,\ V = 0,$

and so there are always two, non-zero eigenvalues, $\lambda = \pm\sqrt{T_n}C$, plus $n-2$ zero eigenvalues.

As regards a non-zero potential energy, there are currently two important physical cases where this occurs:

- The DCE for the URM4 harmonic oscillator solution [3],4
- The DCE for the URM5 relativistic 'mass' solution [3],7, where mass is equated to the potential energy, and the DCE is the relativistic energy-momentum equation per unit mass - see further below.

Looking at the URM3, $n = 3$ case (I19b), this is rearranged as

(I19f) $\lambda^3 = K\lambda + \sqrt{T_n}VC$,

and substituting for $\lambda = \sqrt{T_n}C$, using $T_n^2 = +1$, then dividing throughout by the non-zero eigenvalue C, gives

(I19g) $T_nC^2 = K + V$.

This is a common, quoted form of the DCE in terms of the kinetic and potential energy terms. Note that these are not, in general, the Newtonian forms of kinetic and potential energy. In fact, as regards URM5 (I19d), the kinetic term is related to the relativistic momentum, and the potential energy is related to the rest mass energy, as per the relativistic momentum energy equation – see [3],7 and [4],2-10.

In addition to the characteristic equation, the DCE is also written as the inner product of the zero eigenvector and its reciprocal as in

$$\mathbf{X}^{n0}\mathbf{X}_{n0} = \overline{\mathbf{X}}_{n0}\mathbf{X}_{n0} = (n-2)C^2,\ n \geq 3 \text{ (F11).}$$

(I21) The **Dynamical equations** are the *n* linear equations specified by the single, invariant eigenvector equation

(I21a) $\mathbf{A}_0\mathbf{X}_+ = \sqrt{T_n}C\mathbf{X}_+,\ C > 0,$

which was updated [4] to incorporate the parity operator T_n (I39), and takes the following two values in this particular equation:

$\sqrt{T_n} = +1$, Pythagoras
$\sqrt{T_n} = i$, Skew.

Because the matrix $\mathbf{A}_0$ has a zero lead diagonal, each element, x, y, z etc., of the invariant plus eigenvector $\mathbf{X}_+$ (I11) is a function of only the other $n-1$ coordinates, but not itself, i.e. the coordinates x, y, z etc. do not couple to themselves.

The dynamical equations (I21a) have an equivalent form given in terms of the $\mathbf{A}_+$ matrix and $\mathbf{X}_0$ eigenvector as:

(I21b) $\mathbf{A}_+ \mathbf{X}_0 = -C\mathbf{X}_+$.

(I4) An **Excess dimension** is any dimension higher than the third, i.e. the fourth or higher. In an *n*-dimensional space, an excess dimension r is such that $3 < r \le n$, and its associated temporal (evolutionary) parameter is denoted by t_r [2],C.

(I40) The **Exterior product** (symbol ^) of two, arbitrary, *n*-dimensional vectors is defined in URMT as

$\mathbf{X} \wedge \mathbf{Y} = \mathbf{X}\mathbf{Y}^T - \mathbf{Y}\mathbf{X}^T = \mathbf{X} \otimes \mathbf{Y} - \mathbf{Y} \otimes \mathbf{X}$, the exterior product of **X** and **Y**,

which is the difference of the two, outer vector products $\mathbf{X}\mathbf{Y}^T$ and $\mathbf{Y}\mathbf{X}^T$ (or $\mathbf{X} \otimes \mathbf{Y}$ and $\mathbf{Y} \otimes \mathbf{X}$).

This product is also known as an ‘alternating’, ‘wedge’ or ‘Grassman’ product [5].

See also the Errata online PDF: *http://www.urmt.org/urmt_errata_books1to5.pdf*

In the strict mathematical definition, the exterior product is expanded in terms of basis element known as bi-vectors, whereas the URMT definition above uses matrices. Albeit this is really just a representation issue since URMT can decompose its **A** matrices (formed from exterior products) into elemental matrices, each effectively a bi-vector representation – see [4],6-3 for example.

For more details on the URMT definition and its properties see [4],7-4, particularly [4],7.38 onward.

See [5] for a formal mathematical definition and associated algebra.

(I5) **Flattening** is the term used to describe the eigenvector evolution in URM3 whereby the two eigenvectors $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ align anti-parallel to $\mathbf{X}_{3+}$ as evolution progresses, i.e. as evolutionary time m (or t_3) increases, see Appendix (B) for the URM3 eigenvector evolution equations. As of this book, this flattening behaviour also occurs with the dual,

frequency-domain evolution, whereby the vectors align with $\mathbf{X}_{3-}$ (using URM3 as an example). The vector $\mathbf{X}_{3+}$ itself is **static** (I16) and invariant to arbitrary variations in the time-domain. However, note that $\mathbf{X}_{3+}$ is actually a two-parameter family of integer vectors, parameters k and l (A26), and hence occupies a 2D discrete subspace of 3D; in this sense the 3D flattens to 2D. Because it is an alignment of vectors, the process is also known as 'alignment' herein, but 'flattening' is used exclusively in earlier URM3 literature. That the eigenvectors align anti-parallel, and not parallel, is largely a choice of sign convention.

See also **Compactification** (I32) and [1]#3 for full details.

(I26) The **Hyperboloid** is the set, symbol $\mathbf{H}$, of all URM3 zero eigenvectors $\mathbf{X}_{30}$ (I18), under Pythagoras conditions (I13), which satisfy a hyperbolic equation otherwise known as the **DCE** (I19). This set forms a discrete hyperboloid sheet with a finite, non-zero radius (eigenvalue C, $C > 0$) at the origin. In URMn, using the method of lifting (I6), there is just one unique hyperboloid given by the vector $\mathbf{X}_{0A}$, which is, itself, just an n-dimensional embedded (zero-padded) form of the URM3 eigenvector $\mathbf{X}_{30}$ - see also [2],7 URM4, [2],13 URM5 and Appendix (F), (F10).

The hyperboloid is strictly known as a 'hyperboloid of one sheet' in geometry, and is formed by the rotation of a hyperbola about the *z*-axis (actually the axis is that of the dynamical variable *R* in the third element of the URM3 zero eigenvector $\mathbf{X}_{30}$ (A33b)), and is therefore a surface of revolution, albeit a discrete surface of points.

See [1]#3,4 for a full definition and [3] for a geometric overview.

See also **Cone** (I27) and **Lattice** (I25).

(I36) The **Invariance Principle**

See the main text Section (1), (1.29).

(I37) **Invariant eigenvalue** – see **Variational Methods** (I20).

(I35) The **Invariant Zero Potential** (IZP) solution is a URMT eigenvector solution that has a zero potential energy term in the **DCE** (I19), which remains invariant to all **variations** (I20) of the unity root matrix $\mathbf{A}_0$. The IZP solution is of prime importance to URMT because potential energy is equated with mass, and so a zero potential energy solution represents a massless particle moving at the speed of light, e.g. a photon or graviton.

For URM3, under either **Pythagoras** (I13) or **Skew** (I33) conditions, the potential energy is always zero, so every URM3 solution under these conditions is an IZP. For URM4 and beyond, additional constraints on the dynamical variables are required to give a zero potential. For URM4, the problem was first addressed in [2],5, and then progressed to URM5 later in Section (13) of the same book [2]. This latest book extends URMT, once again, to URM6, for its six-quark representation, Section (13). This URM6 quark solution is also an IZP. URMT does not now currently go beyond 6x6 **A** matrices (I22), i.e. URM6, albeit it is clear that as the order increases, so too do the number of constraints on the dynamical

variables (elements of the unity root matrix $\mathbf{A}_0$) required to retain a zero potential energy term.

The general, *n*-dimensional, IZP solution was first obtained in [2] by a method known as **lifting** (I6), whereby an existing URM3 solution is lifted to higher matrix orders, i.e. URM4 and above. However, the general, lifted solution uses an existing URM3 solution, and cannot embed two, arbitrary, *n*-dimensional vectors, unlike the exterior product method of arbitrary vector embedding (AVE II), [4],5, which can embed two, arbitrary, *n*-dimensional vectors into a general, $(n+1)$-dimensional URMT matrix scheme, and also generate IZPs.

Both the lifting and exterior product methods generate IZPs, which have two non-zero eigenvalues, with all others zero. This combination brings with it three unique types of eigenvector: $\mathbf{X}_+$ and $\mathbf{X}_-$ for the non-zero, positive and negative eigenvalues ($\pm C$), plus a third, composite, zero eigenvector $\mathbf{X}_0$ (I18) formed from a linear combination of all other zero eigenvectors completing the set.

The geometric and physical aspects of an IZP solution are discussed in [4],7-2, with a mathematical discussion on obtaining the particular eigenvalue solution given in [4],5-3.

(I25) The **Lattice** is the union of the upper and lower **cones** (I27) and the **hyperboloid** (I26), formally defined as the union of the sets $\mathbf{C}$ and $\mathbf{H}$, symbol $\mathbf{L}$, where $\mathbf{C}=\mathbf{C}_U\cup\mathbf{C}_L$ (I27) and therefore

$$\mathbf{L}=\mathbf{C}\cup\mathbf{H}.$$

In other words, the lattice represents the complete, *n*-dimensional eigenvector solution. Care has to be exercised in the usage of 'dimension' given here. For example, URM3 comprises two, 2D, discrete cone surfaces and also a single, discrete hyperboloid sheet. All three surfaces (sheets of discrete points) are strictly 2D, but the complete URM3 eigenvector solution is a three-parameter family comprising integer parameters k,l (A26d) and evolutionary parameter t_3 $(\sim m)$ (A28c), hence 'three-dimensional'.

The lattice exhibits numerous geometric properties such as **flattening** (I5) and **Compactification** (I32).

See [1]#3,4 for a full definition and [3],7-2 for a geometric overview.

See also **Compactification** (I32), **Cone** (I27) and **Hyperboloid** (I26).

(I6) **Lifting**, in the context of URMT, is the process of generating eigenvector solutions for an $(n+1)\times(n+1)$ matrix $\mathbf{A}_{n+1}$ using an eigenvector solution to the $n\times n$ matrix $\mathbf{A}_n$, $n\geq 2$. The matrix $\mathbf{A}_n$ is embedded in $\mathbf{A}_{n+1}$, and an eigenvector solution $\mathbf{X}$ to $\mathbf{A}_n$ is also a solution to $\mathbf{A}_{n+1}$, with appropriate zero padding, see [2],7. The matrix $\mathbf{A}_{n+1}$ is usually under **Pythagoras conditions** (I13) with the additional constraint of a zero potential energy.

See [2]7-2 for lifting the URM2 Pythagorean twin (1,1) to form URM3 Pythagorean triples, [2],4 for lifting URM3 to URM4, and [2]13-3 for lifting URM4 to URM5.

(I7) The **magnitude** of a vector is the positive square root of the inner product of a vector with itself, e.g. for $\mathbf{X}_+$ (URM4) then

$|\mathbf{X}_+| = \sqrt{\mathbf{X}_+ \cdot \mathbf{X}_+} = \sqrt{w^2 + x^2 + y^2 + z^2}$, $|\mathbf{X}_+| \geq 0$,

hence $|\mathbf{X}_+| = \sqrt{2}z$ when $w^2 + x^2 + y^2 = z^2$.

Under **Pythagoras conditions** (I13), the magnitude of a URMT eigenvector is always greater than zero, but under skew conditions (I33) the eigenvector has one or more complex elements and the magnitude is zero. See also the **norm** (I8).

(I9) A **Minus eigenvector** $\mathbf{X}_{n-}$, $n \geq 2$, is any vector that satisfies the eigenvector equation $\mathbf{A}\mathbf{X}_{n-} = -C\mathbf{X}_{n-}$ for negative eigenvalue $-C$, matrix $\mathbf{A}$. See also **plus eigenvector** (I11) and **zero eigenvector** (I18). Note that minus eigenvectors are only defined under **Pythagoras** (I13) or **Skew** conditions (I33) because these conditions give a conjugate pair of eigenvalues $\lambda = \pm C$ (Pythagoras) and $\lambda = \pm iC$ (Skew).

(I10) A **Non-trivial eigenvector** is a Pythagorean n-tuple with three or more non-zero elements. For example, a trivial URM3 eigenvector $\mathbf{X}_{3+} = (x, y, z)$ has only two non-zero components, e.g. $(x,0,z)$ $y = 0$, $x, z \neq 0$, where $x^2 = z^2$, i.e. $x = \pm z$ - this is technically then a **Pythagorean double** (I44) In the case of URM4, $\mathbf{X}_{4+} = (w, x, y, z)$, the same trivial vector would be $(0, x, 0, z)$, with $w = 0$ and $y = 0$. A non-trivial URM4 eigenvector would normally be all four elements non-zero, e.g. (2,3,6,7), but it is acceptable to have a URM4, non-trivial eigenvector that comprises only three non-zero elements, e.g. (0,3,4,5), which is just the Pythagorean triple (3,4,5).

(I8) The **norm** of a vector in URMT, using the standard definition of the norm, see [5], is the square root of the inner product of itself with its conjugate, Appendix (E). For example, the norm of $\mathbf{X}_+$, denoted by $\|\mathbf{X}_+\|$, is given by $\|\mathbf{X}_+\| = \sqrt{\mathbf{X}^- \cdot \mathbf{X}_+} = \sqrt{\overline{\mathbf{X}}_+ \mathbf{X}_+}$. Normally the positive square root is assumed unless otherwise stated. If $\mathbf{X}_+$ is an *n*-element Pythagorean n-tuple then the norm is zero since $\mathbf{X}^- \cdot \mathbf{X}_+ = 0$, e.g. (F7). The same remarks also apply to $\mathbf{X}_-$ and its conjugate form $\mathbf{X}^+$, but note that the norm of any one of URM*n*'s, zero eigenvector $\mathbf{X}_{n0j}$, $j = 0 \ldots n-3$, $n>=3$, is non-zero as given by

$$\|\mathbf{X}_{n0j}\| = \sqrt{\mathbf{X}^{n0j} \cdot \mathbf{X}_{n0j}} = +C^2 \text{ (F10).}$$

See also the **magnitude** (I7) and DCE (I19).

(I38) **Normalising to the DCE** is the process by which the zero eigenvectors (I18) are scaled such that their **norm** (I8) is consistent with the DCE (I19), i.e. the inner product of a zero eigenvector $\mathbf{X}_0$ with its conjugate $\overline{\mathbf{X}}_0$ (or $\mathbf{X}^0$) is just the square of the invariant eigenvalue C, to within a sign, as dictated by the **Parity operator** T_n (I39), i.e.

$\overline{\mathbf{X}}_0 \cdot \mathbf{X}_0 = \mathbf{X}^0 \mathbf{X}_0 = T_n C^2$, the DCE (I19), see also (2.42)

(I34) URMT has **no singularity** because it excludes the all-zero, null vector as an eigenvector, i.e. it excludes the origin from the eigenvector solution. This is not just an arbitrary rule or definition, but a genuine algebraic consequence of defining the invariant eigenvalue *C* as non-zero (I21a), which ultimately constrains all eigenvectors to be non-null. Physically speaking, this means that there are no singularities, and the zero point (null vector or origin) is not required in URMT as currently formulated. This is also discussed in [1]#3. Note that *C* cannot be defined as zero because, for example, URM3 already has a zero eigenvalue, and if *C* were also defined as zero then the third eigenvalue would also have to be zero for a zero trace **A** matrix. This would then give three zero eigenvalues and a virtually worthless solution. Hence C is defined as non-zero, and unity or greater when in integers; when in reals it is still defined as non-zero, but is usually the **magnitude** $|\mathbf{X}|$ (I7) of an arbitrary vector **X**, as per AVE I and II, [3] and [4]. Regardless of the actual value of *C*, it will always be greater than or equal to the smallest, positive real value chosen to represent a unit in URMT, i.e. the smallest quantum of measurement. The conditions on *C* are algebraically expressed as

(I34a) $C \in \mathbb{Z} \Rightarrow C \geq 1$

(I34b) $C, |\mathbf{X}| \in \mathbb{R} \Rightarrow C, |\mathbf{X}| > 0$.

(I39) The **Parity operator**, symbol T_n, is simply the scalar, unity magnitude value ± 1

(I39a) $T_n = \pm 1$,

(I39b) $T_n^2 = +1$.

Its value is determined according to whether the **A** matrix (I22) is under **Skew** (I33) or **Pythagoras** (I13) conditions, i.e.

(I39c) $T_n = -1$, Skew conditions

(I39d) $T_n = +1$, Pythagoras conditions.

For more details, see the [4],1.73 onward.

Note that this symbol has not been used in this book because only solutions under Pythagoras conditions are used, i.e. there are no Skew condition solutions.

(I11) A **Plus eigenvector** $\mathbf{X}_+$, e.g. $\mathbf{X}_{n+}$, $n \geq 2$, is any vector that satisfies the eigenvector equation $\mathbf{A}\mathbf{X}_+ = C\mathbf{X}_+$ (I21a) for positive eigenvalue C, $C > 0$, of matrix $\mathbf{A}$. See also **minus eigenvector** (I9) and **zero eigenvector** (I18). Since the eigenvector equation (I21a) is a fundamental definition in URMT, there is always at least one plus eigenvector $\mathbf{X}_+$.

(I12) A **primitive Pythagorean n-tuple** is that which has no common factor in its elements, i.e. all its elements are co-prime (when in integers).

(I13) The **Pythagoras conditions** are a set of relations between the standard and conjugate dynamical variables, i.e. the elements of $\mathbf{A}_n$ URMn, e.g. $\mathbf{A}_3$ URM3 (A1a), and are such

that the eigenvectors of the $n \times n$ matrix $\mathbf{A}_n$, for non-zero eigenvalues, are Pythagorean n-tuples. The unity root matrix $\mathbf{A}_n$ is formed exclusively from the dynamical variables, and the conditions make the matrix skew-symmetric in the first $n-1$ rows and columns, and symmetric in the last row and column. All Pythagoras conditions for URM n include URM $(n-1)$ as a subset.

See [1]#1-15, [1]#2 for URM3, [2],4 for URM4, [2],13 URM5, and this book for URM6, Section (13).

(I44) A **Pythagorean double** is best defined in terms of a Pythagorean triple (I45), i.e. the ordered triple (a,b,c), $0 = a^2 + b^2 - c^2$, but where one, and only one, of the two elements a or b, is zero. For example, if a is zero, then $0 = b^2 - c^2$ so that $c = \pm b$, with the condition that $b \neq 0$. The non-zero, ordered pair (b,c) is then termed a Pythagorean double. Whilst seemingly trivial, it does permit the 2x2 matrix incarnation of URMT, i.e. URM2, to talk in terms of its eigenvectors as Pythagorean doubles, just like URM3's eigenvectors are Pythagorean triples – actually just the **plus** (I11) and **minus** (I9) eigenvectors, since the zero eigenvectors (I18) are known as hyperbolic in URMT because they satisfy the hyperbolic DCE equation (I19), and not Pythagoras. By URMT's method of **lifting** (I6), such URM2, two-element, Pythagorean double eigenvectors can be lifted to URM3 Pythagorean triples, which can then be further lifted to URM4 Pythagorean quadruples etc. - this process is described in [2],7.

(I45) A **Pythagorean triple** comprises any ordered triple (a,b,c), of integers a,b,c that satisfy the Pythagoras equation $0 = a^2 + b^2 - c^2$. This definition is to be interpreted in its loosest sense with the only condition being that $(a,b,c) \neq (0,0,0)$. In other words, a, b and c are allowed to be positive or negative integers, and a may be less than or greater than b; a can be zero, in which case $|b| = |c|$, or b can be zero, in which case $|a| = |c|$ - these 'triples' are then known as **Pythagorean doubles** (I44). Non-primitive triples are also included, i.e. those such that for non-zero, integer factor k, if (a,b,c) is a Pythagorean triple then so too is (ka,kb,kc). Otherwise, primitive solutions are co-prime, i.e. $\gcd(a,b,c) = 1$.

(I14) The **Pythagorean eigenvalues** comprise the set of two, non-zero eigenvalues $\lambda = \pm C$, with all others zero. E.g., for URM4, the four eigenvalues are $\lambda = \pm C$ and $\lambda = 0,0$, i.e. the zero eigenvalue is repeated with a 'multiplicity' [5] of 2. For URM n there are two, non-zero eigenvalues, $\lambda = \pm C$, and the zero eigenvalue is repeated with a multiplicity $n-2$. The specific plus, minus and zero eigenvalues are usually labelled λ_+, λ_- and λ_0 respectively, i.e. $\lambda \in \{\lambda_+, \lambda_-, \lambda_0\}$. They are termed Pythagorean because, under URMT **Pythagoras conditions** (I13), the eigenvectors to the non-zero eigenvalues ($\lambda = \pm C$ here) are Pythagorean n-tuples. Strictly speaking, URMT under Pythagoras conditions only actually mandates at least two non-zero eigenvalues, and the others may not necessarily be zero. For example, in URM4, if the potential is non-zero then complex eigenvalues can arise in place of the zero eigenvalues. In such a case, the complex eigenvectors also satisfy the Pythagoras equation, as do all eigenvectors for non-zero eigenvalues, when under Pythagoras conditions. Note that the exterior product formulation, [4],7 and [4],8, always gives Pythagorean eigenvalues and eigenvectors when the parity operator (I39) is set to unity, i.e. $T_n = +1$.

See [4],5-3 and the IZP, (I35) or [4],7-2, for more information on this Pythagorean eigenvalue combination.

(I15) **Scale factors**, see **Divisibility factors** (I2).

(I33) The **Skew Conditions** are a set of relations between the standard and conjugate dynamical variables in URM n that make the **A** matrix skew-symmetric (or anti-symmetric), i.e.

(I33a) $\mathbf{A} = -\mathbf{A}^T$, Skew or anti-symmetric.

The conditions are such that the eigenvectors of an $n \times n$ matrix $\mathbf{A}_n$, for non-zero eigenvalues, have both a zero **magnitude** (I7) and **norm** (I8), in contrast to the **Pythagoras conditions** (I13), where the eigenvectors have a zero norm but non-zero magnitude. The eigenvectors of a skew matrix $\mathbf{A}$, for non-zero eigenvalues, are complex, and hence skew conditions represent the complex 'flavour' of URMT. Note that the dynamical variables, i.e. the elements of $\mathbf{A}$, are not necessarily complex and, in fact, they are usually only complex after variational methods are applied.

(I16) A **Static** quantity in URMT is any quantity (invariably an eigenvector or matrix) not dependent on any evolutionary time parameter *t* or t_j, $j = 3 \ldots n$ (time-domain evolution) or frequency f_j (frequency domain evolution) – see the main text, Section (1-7). The plus eigenvector $\mathbf{X}_{n+}$ (I11), $n \geq 2$ (this includes URM2), is the classic URMT example of a static eigenvector when under time-domain evolution, and $\mathbf{X}_{n-}$ when under frequency domain evolution.

(I43) The **Sum set** of eigenvectors $\{\mathbf{X}_{S+}, \mathbf{X}_{S0}, \mathbf{X}_{S-}\}$ and **A** matrices (I22) $\{\mathbf{A}_{S+}, \mathbf{A}_{S0}, \mathbf{A}_{S-}\}$ is the *z*-axis equivalent of the *x*-axis **Base** set (I41) and *y*-axis **Difference** (I42) sets.

(I28) The **T Operator** is an $n \times n$ ($n \geq 2$) matrix defined in two different forms, Pythagoras or Skew, as follows:

(I28a) $\mathbf{T} = \begin{pmatrix} \mathbf{I}_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$ Pythagoras

(I28b) $\mathbf{T} = -\mathbf{I}_n$ Skew,

where $\mathbf{I}_n$ is the identity matrix order n, $n \geq 2$.

As of [4], the **T** operator is also written in the more general form in terms of the parity operator T_n (I39) as

$$\mathbf{T}_n = \begin{pmatrix} T_n \mathbf{I}_{n-1} & 0 \\ 0 & -1 \end{pmatrix}.$$

In both Pythagoras and skew forms, **T** is all zero except for the lead diagonal, which comprises plus or minus unity elements only, i.e. +/-1.

The Pythagoras form is the same as the Cartesian, Minkowski metric in Special Relativity [17], disregarding sign convention.

The main use for **T** is to convert between standard and conjugate eigenvectors, see Appendix (E).

The Pythagoras **T** operator (I28a) is n-dimensional, and the presence of the $n-1$ order identity matrix $\mathbf{I}_{n-1}$ in its upper left means that the sign of each of the first $n-1$ elements of $\mathbf{x}$ remains the same, whilst only the last, nth vector element changes sign when operated on (multiplied) by **T**. Conversely, the skew form of **T** flips the sign of all n elements.

Note that the **T** operator also has the following useful properties:

(I28c) $\mathbf{T}^2 = \mathbf{I}$, $\mathbf{T}^{-1} = \mathbf{T}$, $\mathbf{T}^T = \mathbf{T}$.

Strictly speaking, each form of **T** splits into two more forms, standard and reciprocal:

$\mathbf{T}_n$ standard $n \times n$ form

$\mathbf{T}^n$ reciprocal $n \times n$ form.

However, they are both identical, i.e.

$\mathbf{T}^n = \mathbf{T}_n$.

(I31) A **Unity Root Matrix**, denoted by generic symbol $\mathbf{A}_n$ (or $\mathbf{A}_0$, $\mathbf{A}_{n0}$), is defined as an integer, $n \times n$ matrix with an all-zero lead diagonal, that is symmetric in its elements (known as dynamical variables) to within URMT conjugation (conjugate relations, e.g. (A2g) to (A2i)), where all the off-diagonal elements satisfy unity root or power-residue definitions (A2). See the founding, URM3 matrix $\mathbf{A}_3$ (A1a), and [1]#1 for full information, in particular, [1]#1, Appendix (C1) for a cubic example.

(I24) **Variational Matrices**

See annihilator matrices (I23) and variational methods (I20) next.

(I20) **Variational Methods**

Any variation to the fundamental **A** matrix, (A1a) or its *n*-dimensional variant, such that its fundamental defining equation

(I20a) $\mathbf{A}\mathbf{X}_+ = C\mathbf{X}_+$, $\lambda = C$, $C > 0$,

remains invariant to the variation, is termed a variational method in URMT. Applying this method (variation) leaves both the **plus eigenvector** $\mathbf{X}_+$ (I11) and the eigenvalue C invariant, but invariably modifies ('evolves') all other eigenvectors of the **A** matrix.

Such variational methods are achieved by adding an **annihilator** matrix Δ (I23) to **A**, where Δ can be absolutely any matrix that satisfies its annihilation definition $\Delta\mathbf{X}=0$ (I23a). Usually this annihilator is the 'plus' **A** matrix '$\mathbf{A}_+$', and the vector **X** is $\mathbf{X}_+$, since $\mathbf{A}_+\mathbf{X}_+ = 0$. Conversely, in the dual URMT formulation, the annihilator matrix is $\mathbf{A}_-$, and **X** is $\mathbf{X}_-$, since $\mathbf{A}_-\mathbf{X}_- = 0$. Whilst these two matrices, $\mathbf{A}_+$ and $\mathbf{A}_-$, are commonly used as annihilators of $\mathbf{X}_+$ and $\mathbf{X}_-$ respectively, they are not the only annihilator matrices, and Δ is generally arbitrary so long as it satisfies its definitions.

There are two types of variation, global and local, where each type changes the dynamical variables in the **A** matrix. A global change affects every dynamical variable in the same way, and is the same as adding (or subtracting) a scalar multiple m of Δ to **A** as in the following mapping:

(I20b) $\mathbf{A} \to \mathbf{A} - m\boldsymbol{\Delta}$.

By the annihilator definition, $\boldsymbol{\Delta}\mathbf{X} = 0$ (I23), this transformation on **A** leaves the eigenvector $\mathbf{X}$ invariant as follows:

(I20c) $(\mathbf{A} - m\boldsymbol{\Delta})\mathbf{X} = \mathbf{A}\mathbf{X} - m\boldsymbol{\Delta}\mathbf{X} = \mathbf{A}\mathbf{X} = C\mathbf{X}$, $\lambda = C$.

Whilst a global variation involves just a single parameter, e.g. m in (I20b), local variations act internally on Δ, and there are far more possible variations satisfying the same property (I20c). See [1]#1 and [2],8 for full details.

(I17) **Vector space, eigenvector space**

These two interchangeable terms are used loosely throughout URMT since they refer to the eigenvectors of the unity root matrix **A** (A1a), but not linear combinations of them, i.e. the whole of URMT currently studies the vector basis comprising the eigenvectors, but not arbitrary vectors generated from them. Thus, for the purists, the vector space would be better described as an infinite set (space) of linearly independent eigenvectors, rather than 'vector space' in the true, mathematical sense of the word. See [5] for a strict definition of a vector space. For instance, the URMT discrete, vector space is not generally closed and neither is there a null vector, i.e. a vector of all zeroes, see also *no singularity* (I34).

The URM3 set of eigenvectors is defined by the infinite set of points termed the **lattice** (I25). The key point is that every vector in the lattice is an eigenvector of the unity root matrix, and arbitrary, linear combinations of eigenvectors do not generally give another eigenvector in the lattice, as is true for any general set of eigenvectors for distinct eigenvalues. For example, the eigenvector sum $\mathbf{X}_{3+} + \mathbf{X}_{3-}$ is not an eigenvector of $\mathbf{A}_{30}$ even though $\mathbf{X}_{3+}$ and $\mathbf{X}_{3-}$ are both eigenvectors, eigenvalues C and $-C$ respectively, Appendix (A). That is not to say that the n URMT eigenvectors cannot form the basis of an n-dimensional vector space - indeed they can by their linear independence, which is why the term 'vector space' is used loosely.

Currently, however, URMT focuses solely on the eigenvectors themselves but not arbitrary functions of them.

(I18) A **Zero Eigenvector** in URMT is defined as an eigenvector $\mathbf{X}_{n0}$ satisfying $\mathbf{A}\mathbf{X}_{n0} = 0$, i.e. an eigenvector for eigenvalue zero. In linear algebra, such vectors are said to occupy the null or kernel space of the matrix **A**. The URMT eigenvectors $\mathbf{X}_0$, $\mathbf{X}_{30}$, $\mathbf{X}_{n0A}$, $\mathbf{X}_{n0B}$, $\mathbf{X}_{n0C}$, $\mathbf{X}_{n0j}$, $j = 0 \ldots n-3$ etc., are all zero eigenvectors since they are the eigenvectors for the repeated, zero **Pythagorean eigenvalue** (I14). Generally, zero eigenvectors are only considered under Pythagoras or Skew conditions, e.g. (A19) for URM3 Pythagoras.