From unity root matrix theory to special relativity

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Abstract. This paper is a review of some key methods and results arising from the first few years of original research in the new subject of Unity Root Matrix Theory, which the author believes may offer a discrete formulation of physical laws at the most fundamental, Planck level. This belief stemming from the similarity of the methods used, principles employed, and results obtained, to mathematical physics. The paper starts with a comprehensive review of unity root matrix theory's founding, 3x3 formulation and the physical phenomena that led to its development towards a discrete description of the laws of nature. It then proceeds to the higher-dimensional extensions with 5x5 matrices and two related examples in the Special Theory of Relativity. The first example is a Doppler-parameterised solution showing cosmological expansion in accordance with the Hubble law, and the second example illustrates the implicit introduction of mass via a non-zero scalar potential in the energy conservation equation. The paper completes by converting the second solution back to a Newtonian form.

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Acronyms and Abbreviations

DCE : Dynamical Conservation Equation
SPI : Standard Physical Interpretation
STR : Special Theory of Relativity
URMT : Unity Root Matrix Theory
URMn : the $n \times n$ matrix formulation of URMT, $n \geq 2$.

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1 Introduction - why URMT?

Unity Root Matrix Theory's origins lie in number theory and the study of nth order Diophantine equations and congruence relations in their variables [1]. This early work resulted in an eigenvector equation to an all-integer, unity root matrix, where the characteristic equation is treated as an energy conservation equation. By applying novel variational techniques to this equation, with an associated invariance principle, an invariant, eigenvector solution can be obtained, which forms the foundations of URMT. Further simplifications give a complete eigenvector solution that has a consistent, physical, dynamical interpretation, with acceleration, velocity and position eigenvectors related by classical calculus relations, and conservation laws with scalar invariants arising between the eigenvector inner products. Extending the formulation to four and higher dimensions [2], with the invariant eigenvalue attributed to the speed of light, gives a relativistic, Doppler solution [3]. Extending the solution further shows that the potential energy can be related to the appearance of mass via a sub-luminal velocity, with the characteristic equation identical to the relativistic energy-momentum equation. Lastly, this relativistic mass solution can be simplified to show a classical, Newtonian, constant acceleration solution in its first three dimensions.

Given the tremendous success of the Standard Model and Relativity, it is worth explaining just why there is any need for a new theory, or at least a reformulation of existing physics in a discrete form.

First and foremost, it is the belief of the author that, at the Planck level, nature is just not as complicated as currently formulated with regard to the current theories of a unified quantum gravity. The reasoning behind this belief in a simpler formulation stems directly from the fact that nature may well appear complex, but never really is when its workings are revealed. Witness the myriad of life that arises from just four DNA bases, or the apparent complexity of patterns in nature, the latter arising from the repeated application of a simple rule. Related to this is the highly unpredictable, chaotic phenomena that materialises from a repeated folding action. See [4] for examples of chaotic and complex phenomena. It is no coincidence that the process of folding is akin to modular arithmetic, and fundamental to URMT. In particular, in its definition of unity roots (or primitive roots [5]) and its variational methods. The same repeated application of modular arithmetic is also commonly used in the generation of random number sequences, i.e. what appears random has, in fact, a very simple rule at its heart. In brief, it is thought all the laws of nature must ultimately have this simplicity.

This belief is also behind the reason why a discrete formulation of nature is considered necessary, and that we cannot keep applying our continuous differential equations at ever smaller levels. For example, URMT shows a natural calculus can arise even without any preconceived notions of limits or real analysis, it just emerges from the laws of large numbers. Whereas we tend to think of approximating differential equations in a discrete form using a computer, in fact we are really approximating nature by using such continuous forms in the first place, and we would be better off starting from the bottom upward with a discrete theory, and ensuring its large limit is that of the differential equations of mathematical physics.

Whilst this might seem speculative, the results of URMT, many of which are reviewed in this paper, show a lot of physics can come from what are rather abstract origins in number theory.

Developing a discrete theory like URMT is not to say the Standard Model or Relativity are wrong - far from it. Everyday they seem to get further supporting evidence, and are believed to be correct precisely because their formulation, based on the principles of invariance and symmetry, is correct at the macroscopic level, i.e. many orders above the Planck scale. Note that any elemental formulation of nature does not necessarily have to be at the Planck scale, i.e. $O(10^{-35})\,m$ and $O(10^{-44})\,s$. However, whatever its scale, it will almost certainly materialise many orders of magnitude smaller than that currently observable, and perhaps even smaller than the Planck scale. The simple reasoning for this statement is that currently the macroscopic, observable world (spacetime anyhow) appears continuous, with such perceived continuity being attributed, in a discrete theory, to the extremely large numbers involved.

Whilst the Standard Model and Relativity are considered correct, we do not have a unified theory of quantum gravity, neither does the author believe we will get one by continuing with a continuous theory of nature. In brief, it is not the general principles that are considered wrong, merely the formulation at the most elemental level.

Most URMT work to date reproduces existing physics with some predictions such as cosmological expansion [3] and compactification [2], but it is accepted that a new discrete formulation should also produce new predictions. Indeed, it was hoped that, given URMT's invariant eigenvalue can be consistently associated with the speed of light (c), it might enable its value to be calculated directly. However, so far, it is an input constant (the invariant eigenvalue), and URMT can be made to work with an arbitrary value of c - unity preferred (hence unity roots). Nevertheless, that c can even be successfully associated with equations in number theory, with scant regard to physics, lends further credence to URMT as a physical theory. With this in mind, the determination of the gravitational constant (G), and or Planck's constant (h), may well be possible. But even if URMT were just another formulation, the Hamiltonian formulation of mechanics is testimony to how such a reformulation of existing laws and methods, i.e. Lagrangian dynamics, can eventually lead to new physics, i.e. quantum mechanics.

To finish this justification, the biggest reason to pursue URMT is that, after a lot of work in the subject, "*it has far too many similarities to mathematical physics to ignore*", and given we seem to be no nearer to a unified theory, thirty or more years after what might be considered the most successful theory of recent times, i.e. Electroweak, it is not time to restrict ourselves to just one or two candidates, but also consider some more radical alternatives.

The nearest related subject to URMT is probably discrete physics. But URMT is not about converting existing continuous equations into a discrete form, but rather it starts with integer (Diophantine) equations and then proceeds to develop their physical associations. As such, it appears to be a completely new subject area (less than five years old) and, to the author's best knowledge, the only currently available texts are the first three books published, [1], [2] and [3], plus some free material at the web-site http://www.urmt.org including a lengthy presentation and overview - follow the link 'free PDF Downloads'.

2 Unity Root Matrix Theory 3x3

This section is a review of the original 3x3, 'URM3' formulation of URMT, first published in [1].

2.1 Origins in Number Theory

The number-theoretic origins of URMT lead to the following three linear equations in three, integer unknowns x, y, z, for positive, integer constant C (soon to become an eigenvalue), and unity roots P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ - strictly speaking these are only 'unity' roots (or primitive roots) when $C = 1$, and power residues otherwise for $C > 1$; see [5].

$$Cx = Ry + \overline{Q}z \tag{2.1}$$

$$Cy = \overline{R}x + Pz \tag{2.2}$$

$$Cz = Qx + \overline{P}y \tag{2.3}$$

$$C \in \mathbb{Z},\ C \geq 1, \tag{2.4}$$

$$x, y, z \in \mathbb{Z},\ (x, y, z) \neq (0,0,0),\ \gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1 .$$

$$P^n \equiv +C^n \pmod{x},\ Q^n \equiv +C^n \pmod{y},\ \ R^n \equiv -C^n \pmod{z},\ n \in \mathbb{Z},\ n \geq 2$$

$$\overline{P}^n \equiv +C^n \pmod{x},\ \overline{Q}^n \equiv +C^n \pmod{y},\ \overline{R}^n \equiv -C^n \pmod{z} \tag{2.5}$$

$$P, Q, R \in \mathbb{Z},\ (P, Q, R) \neq (0,0,0),\ \overline{P}, \overline{Q}, \overline{R} \in \mathbb{Z},\ (\overline{P}, \overline{Q}, \overline{R}) \neq (0,0,0) .$$

From a physical perspective, the three equations (2.1) to (2.3) can be thought of as representing three, coupled objects (x, y, z), each one couples to the other two, but not itself, no one object is more important than the other. This ternary aspect, and nature's own abundance of groupings of three e.g. three generations of particles and three spatial dimensions, plus a notion of conjugacy amongst pairs of variables, i.e. particle/anti-particle, provided some early impetus to pursue possible physical links.

The unity roots P, Q, R are related to their conjugates $\overline{P}, \overline{Q}, \overline{R}$ by the following 'conjugate relations':

$$C^{n-2}\overline{P} \equiv P^{n-1} \pmod{x},\ C^{n-2}\overline{Q} \equiv Q^{n-1} \pmod{y},\ C^{n-2}\overline{R} \equiv -R^{n-1} \pmod{z} \tag{2.6}$$

The three variables x, y, z satisfy the following 'coordinate equation' [1] for some integer k :

$$0 = x^n + y^n - z^n + xyz.k(x, y, z),\ k \in \mathbb{Z},\ k \neq 0 .$$

Of course, k is never zero for $n > 2$, as proven by Wiles [6].

The following scale (or divisibility) factors α, β and γ are defined in terms of x, y, z, the unity roots (2.5) and constant C (2.4) as follows, they are also all integers:

$$
\begin{aligned}
(C^2 - P\overline{P}) &= \alpha x \\
(C^2 - Q\overline{Q}) &= \beta y \\
(C^2 - R\overline{R}) &= \gamma z
\end{aligned}
\tag{2.7}
$$

$$\alpha, \beta, \gamma \in \mathbb{Z},\ (\alpha, \beta, \gamma) \neq (0,0,0).$$

The above definitions derive from the number-theory side of URMT. However, given the real interest of the author and, in fact, the entire subject of URMT is 'Physics in Integers' [1], then a more physical derivation will be given shortly in terms of a single energy equation, subjected to variational methods, and coupled with an invariance principle.

2.2 The Unity Root Matrix and Invariant Eigenvector

Defining the unity root matrix $\mathbf{A}_3$ and vector $\mathbf{X}_{3+}$ in terms of the coordinates x, y, z and unity roots P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ as follows:

$$\mathbf{A}_3 = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix},\ \mathbf{X}_{3+} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.8}$$

then the three linear equations (2.1) to (2.3) are written in matrix form, eigenvalue C, as

$$\mathbf{A}_3 \mathbf{X}_{3+} = C\mathbf{X}_{3+} \tag{2.9}$$

The linear equations represented by (2.9) are known as the 'dynamical equations' in URMT due to the later association of the unity roots as velocity quantities; see the standard physical interpretation (SPI), Section (2.15). Likewise, the unity roots P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ are termed 'dynamical variables'.

When C is unity, the matrix $\mathbf{A}_3$ is a true unity root matrix, and when $C > 1$ the congruences (2.5) are non-unity, power residues. However, [1] shows how the theory for $C > 1$ can be developed assuming unity roots, i.e. $C = 1$, and the matrix $\mathbf{A}_3$ is termed a unity root matrix regardless of the value of C - unity or greater.

A reciprocal, row eigenvector is defined in terms of the scale factors α, β, γ (2.7) as

$$\mathbf{X}^{3+} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}, \tag{2.10}$$

and satisfies the eigenvector equation

$$\mathbf{X}^{3+} \mathbf{A}_3 = C\mathbf{X}^{3+}.$$

2.3 The Dynamical Conservation Equation

To obtain eigenvector solutions to the dynamical equations (2.9), the non-singular determinant condition on $\mathbf{A}_3$, for eigenvalue λ, is

$$\det(\mathbf{A}_3 - \lambda I_3) = 0\,,$$

which expands to the following characteristic equation:

$$0 = -\lambda^3 + (P\overline{P} + Q\overline{Q} + R\overline{R})\lambda + (PQR + \overline{P}\overline{Q}\overline{R})\,. \tag{2.11}$$

Of course, one such eigenvector $\mathbf{X}_{3+}$, eigenvalue C, is already given (2.8). However, its analytic solution in terms of the dynamical variables (unity roots) will be derived again in Section (2.6).

Defining the kinetic energy term K and potential energy term V (per unit mass) as follows

$$K = P\overline{P} + Q\overline{Q} + R\overline{R}\text{ , kinetic energy per unit mass,} \tag{2.12}$$

$$V = (PQR + \overline{P}\overline{Q}\overline{R})/C\text{ , potential energy per unit mass} \tag{2.13}$$

then (2.11) shortens to

$$0 = -\lambda^3 + K\lambda + VC\,. \tag{2.14}$$

Substituting for the single eigenvalue $\lambda = C$ and dividing throughout by C, which is always positive, non-zero by (2.4), then the following 'Dynamical Conservation Equation' is obtained

$$C^2 = K + V\text{ , the Dynamical Conservation Equation (the DCE).} \tag{2.15}$$

This will form the founding equation for the physical development of the theory. The justification for ascribing the terms in the dynamical variables as velocity quantities is seen throughout URMT by the consistent physical results it gives.

Written in full, using K (2.12) and V (2.13), the DCE is

$$C^2 = (P\overline{P} + Q\overline{Q} + R\overline{R}) + \frac{(PQR + \overline{P}\overline{Q}\overline{R})}{C}\,. \tag{2.16}$$

Barring the explicit omission of mass, it can already be seen where URMT is heading with (2.15), i.e. multiplying by the mass m gives

$$E = mC^2 = m(K + V)\text{ , } K \text{ and } V \text{ are per unit mass.,} \tag{2.17}$$

This energy equation pervades all URMT, as will be seen throughout this paper.

2.4 The Potential Equation

Summing all three divisibility relations (2.7) gives

$$3C^2 - (P\overline{P} + Q\overline{Q} + R\overline{R}) = \alpha x + \beta y + \gamma z\,,$$

and by substituting for the kinetic energy K (2.12), and using the DCE (2.15), this becomes the 'potential equation':

$$2C^2 = \alpha x + \beta y + \gamma z - V \text{, the potential equation.} \tag{2.18}$$

This is another conservation equation in URMT, and links all four sets of variables x, y, z, P, Q, R, $\overline{P}, \overline{Q}, \overline{R}$ and α, β, γ (2.7) to give another invariant $2C^2$ to go with C^2 (the total energy per unit mass) in the DCE (2.15).

Some physical justification for calling this the potential equation and, indeed, assigning the name potential in (2.13), can be seen by noting that the scale factors α, β, γ are later physically associated with position coordinates, and the coordinates x, y, z as accelerations - or rather negative force per unit mass; see Section (2.15). With this in mind, by defining a gradient operator as

$$\nabla = (\partial_\alpha, \partial_\beta, \partial_\gamma), \tag{2.19}$$

then applying this to the potential equation (2.18), for constant C (invariant by definition), gives

$$\nabla V = (x, y, z),$$

and since x, y, z are associated with accelerations, i.e. force per unit mass ($\mathbf{F}$), then ∇V is consistent with the standard force/potential equation

$$\mathbf{F} = -\nabla V = -(x, y, z) \text{, force per unit mass.}$$

Although this form of gradient operator (2.19) is not actually used further, the consistent derivation here of a force from a scalar potential provides some more substance to the rather abstract energy terms and physical associations mentioned above and detailed in Section (2.15).

2.5 Variational Methods

Local variations

Because the dynamical variables are unity roots, defined by congruences (2.5), they are not unique, and their definitions hold true for three, arbitrary, integer 'local' variations $\eta, \delta, \varepsilon$

$$\begin{gathered} P \to P + \delta x,\ \overline{P} \to \overline{P} - \varepsilon x, \\ Q \to Q + \varepsilon y,\ \overline{Q} \to \overline{Q} - \eta y, \\ R \to R + \eta z,\ \overline{R} \to \overline{R} - \delta z, \\ \eta, \delta, \varepsilon \in \mathbb{Z}. \end{gathered} \tag{2.20}$$

The variations are termed local because they act independently on each row of the unity root matrix $\mathbf{A}_3$ (2.8), e.g. R and $\overline{Q}$ in the top row are varied by η. Conversely, a single 'global delta variation', symbol δ, will be defined later (2.26) that acts on the entire matrix $\mathbf{A}_3$.

In matrix form, these local variations are represented by the following variational matrix, symbol $\mathbf{\Delta}$:

$$\mathbf{\Delta} = \begin{pmatrix} 0 & +\eta z & -\eta y \\ -\delta z & 0 & +\delta x \\ +\varepsilon y & -\varepsilon x & 0 \end{pmatrix}. \tag{2.21}$$

This matrix has the following annihilator property:

$$\mathbf{\Delta X}_{3+} = 0. \tag{2.22}$$

The variation on the unity root matrix $\mathbf{A}_3$ is written as the mapping

$$\mathbf{A}_3 \to \mathbf{A}_3 + \mathbf{\Delta},$$

and, by design, the transformation $\mathbf{A}_3 + \mathbf{\Delta}$ leaves $\mathbf{X}_{3+}$ invariant as follows, using (2.22),

$$(\mathbf{A}_3 + \mathbf{\Delta})\mathbf{X}_{3+} = \mathbf{A}_3\mathbf{X}_{3+} + \mathbf{\Delta X}_{3+} = \mathbf{A}_3\mathbf{X}_{3+} = C\mathbf{X}_{3+}.$$

Thus, using (2.21), the most general form of $\mathbf{A}_3$ that preserves the eigenvector equation (2.9) is

$$\mathbf{A}_3 = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix} + \begin{pmatrix} 0 & +\eta z & -\eta y \\ -\delta z & 0 & +\delta x \\ +\varepsilon y & -\varepsilon x & 0 \end{pmatrix}.$$

By comparing this matrix $\mathbf{A}_3$ with the original, unvaried form (2.8), the transformation equations in the dynamical variables (2.20) are verified.

Whether the variation is local or global, all equations in the dynamical variables, plus the eigenvector $\mathbf{X}_{3+}$ and eigenvalue C, remain invariant to these variations, and this is enshrined in URMT in the following Invariance Principle:

2.6 The Invariance Principle

The dynamical equations and their solutions are invariant to a coordinate transformation in the dynamical variables.

The coordinate transformation in the dynamical variables is already defined above as the local variations (2.20). By applying these local transformations to the DCE (2.16), six separate terms, one for each of the six possible combinations of the local variational elements $\eta, \delta, \varepsilon$, are obtained. Collecting the three separate, quadratic variations in $\delta\varepsilon$, $\eta\varepsilon$ and $\eta\delta$, and equating to zero since the equations must remain invariant by the above principle, then the same three dynamical equations, as first given in (2.2), are obtained, i.e.

$$Cx = Ry + \overline{Q}z\text{ , } \delta\varepsilon \text{ term (2.1)}$$

$$Cy = \overline{R}x + Pz\text{ , } \eta\varepsilon \text{ term (2.2)}$$

$$Cz = Qx + \overline{P}y \text{, } \eta\delta \text{ terms (2.3).}$$

However, the remaining three, linear (first degree) variational terms in $\eta, \delta, \varepsilon$ give the three possible solutions in x, y, z to these dynamical equations, two of which are independent:

$$z(C\overline{R} + PQ) = y(CQ + \overline{R}\overline{P}) \text{, } \eta \text{ term, } z \text{ in terms of } y \text{,}$$

$$z(CR + \overline{P}\overline{Q}) = x(C\overline{P} + QR) \text{, } \delta \text{ term, } z \text{ in terms of } x \text{,} \tag{2.23}$$

$$y(C\overline{Q} + RP) = x(CP + \overline{Q}\overline{R}) \text{, } \varepsilon \text{ term, } y \text{ in terms of } x \text{.} \tag{2.24}$$

A key feature of this URMT-unique variational method is that, when applied to its foundation, energy conservation equation (the DCE), it enables both the dynamical equations and their solutions to be obtained. This application of a variational method to an energy conservation equation is considered analogous to an action principle applied to a Lagrangian function to obtain the field equations. However, the Lagrangian method only provides equations that then require solving, which is not the case here, where they effectively come for free.

One possible eigenvector solution for $\mathbf{X}_{3+}$ can be obtained from (2.23) and (2.24) in terms of x as a parameter:

$$\mathbf{X}_{3+} = x.\begin{pmatrix} 1 \\ \left(\dfrac{CP + \overline{Q}\overline{R}}{C\overline{Q} + RP}\right) \\ \left(\dfrac{C\overline{P} + QR}{CR + \overline{P}\overline{Q}}\right) \end{pmatrix}. \tag{2.25}$$

See [1], paper 5, Appendix (C) for a full list of the nine possible forms of eigenvector solution (three of which are linearly independent) albeit they are given for a unity eigenvalue $C = 1$. Nevertheless, it is very easy to convert them to a general eigenvalue C simply by ensuring every first degree term in a dynamical variable becomes a second degree term by multiplication by C, e.g. a term such as $P + \overline{Q}\overline{R}$ becomes $CP + \overline{Q}\overline{R}$. Thus all terms are of second degree in the dynamical variables (including the eigenvalue C), i.e. velocity squared, or energy per unit mass; Section (2.15).

The reader is reminded that all the above equations in this section are valid for a general eigenvalue λ. If all three eigenvalues are known then all three eigenvectors can be obtained simply by replacing C with one of the three possible values for λ, which are the three roots of the cubic polynomial (2.11). So far, however, only a single eigenvector $\mathbf{X}_{3+}$, eigenvalue C, has been obtained.

The analytic form (2.25) of the eigenvector $\mathbf{X}_{3+}$, eigenvalue $\lambda = C$, is not actually required further, and its general form $\mathbf{X}_{3+}$ (2.8) is sufficient to proceed. A more amenable, parametric form is given further below in Section (2.12).

2.7 A Global Pythagoras Variation

Returning to the variational methods in Section (2.5), by setting the two, local variational parameters η and ε to the value of δ in the following way

$$\eta = \delta\,,\ \varepsilon = -\delta\,, \tag{2.26}$$

then the variational matrix $\mathbf{\Delta}$ (2.21) simplifies to the following, 'Pythagorean' form, symbol $\mathbf{\Delta}^P$:

$$\mathbf{\Delta}^P = \delta\begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ -y & +x & 0 \end{pmatrix}. \tag{2.27}$$

The reason for this Pythagorean nomenclature will be seen shortly in the next section where vectors $\mathbf{X}_{3+}$ (2.8) and $\mathbf{X}^{3+}$ (2.10) are seen to satisfy the Pythagoras equation. By (2.20), the dynamical variables also now transform as follows:

$$P \to P + \delta x\,,\ Q \to Q - \delta y\,,\ R \to R + \delta z\,,$$
$$\overline{P} \to \overline{P} - \delta x,\ \overline{Q} \to \overline{Q} - \delta y\,,\ \overline{R} \to \overline{R} - \delta z\,,$$

and substituting for these transformed dynamical variables into the DCE (2.16), collecting terms in δ and δ^2, and equating to zero in accordance with the Invariance Principle, then the following two expressions are obtained:

$$\delta\text{ term:}\quad \begin{aligned} 0 &= x\left(\frac{QR + \overline{Q}\overline{R}}{C}\right) - y\left(\frac{RP + \overline{R}\overline{P}}{C}\right) + z\left(\frac{PQ - \overline{P}\overline{Q}}{C}\right). \\ &+ x(\overline{P} + P) - y(\overline{Q} + Q) + z(\overline{R} - R) \end{aligned} \tag{2.28}$$

$$\delta^2\text{term : } 0 = x^2 + y^2 - z^2 - xy\left(\frac{R + \overline{R}}{C}\right) + xz\left(\frac{Q - \overline{Q}}{C}\right) + yz\left(\frac{\overline{P} - P}{C}\right). \tag{2.29}$$

The second, δ^2 term is noteworthy in that nowhere, so far, has a quadratic exponent been asserted, and this variation holds for all exponents $n \geq 2$, not just $n = 2$, yet from this nth order derivation the Pythagoras equation naturally emerges.

2.8 URM3 Pythagoras Conditions

It can be seen from the δ^2 (2.29) that if the conjugate dynamical variables $\overline{P}, \overline{Q}, \overline{R}$ are equated to their standard forms P, Q, R as follows:

$$\overline{P} = P\,,\ \overline{Q} = Q\,,\ \overline{R} = -R\,,\ \text{the Pythagoras conditions,} \tag{2.30}$$

then (2.29) reduces to the Pythagoras equation, i.e.

$$0 = x^2 + y^2 - z^2 . \tag{2.31}$$

This means that the elements x, y, z of eigenvector $\mathbf{X}_{3+}$ also satisfy the Pythagoras equation. Whilst a quadratic, Pythagorean exponent $n = 2$ has not been asserted, by enforcing the Pythagoras conditions (2.30) on the dynamical variables, if the coordinates satisfy Pythagoras (2.31) then they cannot simultaneously satisfy a higher order, $n > 2$, form of the coordinate equation (2.7), and the k-value must therefore be zero in this case.

Although not demonstrated here (see (2.54)), the scale factors α, β, γ also satisfy Pythagoras, i.e.

$$0 = \alpha^2 + \beta^2 - \gamma^2 . \tag{2.32}$$

Congruence relations such as $\overline{P} \equiv P(\operatorname{mod} x)$ can be obtained by substituting $n = 2$ in the conjugate relations (2.6), but the Pythagoras conditions (2.30) are actually equalities (identities), and not congruences. If P and $\overline{P}$ are congruent, but not identical, then the coordinate equation (2.7) remains perfectly valid as a quadratic Diophantine equation, except its k-value is now non-zero.

If the above was all there was to URMT then, whilst intriguing, it might remain a curio with not much real physics. However, armed with the Pythagoras conditions (2.30) and equations of quadratic degree, this is where most of the real URMT physics starts.

2.9 An Invariant Zero Potential

Applying the Pythagoras conditions (2.30) to the kinetic term K (2.12) and potential term V (2.13), they become

$$K = P^2 + Q^2 - R^2 , \tag{2.33}$$

$$V = 0 . \tag{2.34}$$

The Potential energy is thus zero and the DCE (2.15) simply becomes constant energy, kinetic term:

$$C^2 = K \text{, the DCE, energy per unit mass,} \tag{2.35}$$

and implies by (2.33) that the dynamical variables P, Q, R satisfy the following hyperbolic conservation equation, which is, once again, a form of the DCE.

$$C^2 = P^2 + Q^2 - R^2 . \tag{2.36}$$

With $V = 0$, and under Pythagoras conditions (2.30), the potential equation (2.18) reduces to

$$2C^2 = \alpha x + \beta y + \gamma z \text{, the potential equation.} \tag{2.37}$$

The characteristic equation (2.14) also simplifies to

$$0 = -\lambda^3 + K\lambda ,$$

and thus, using K (2.35), there are three, symmetric eigenvalues

$$\lambda = \pm C, 0\,.$$

By applying the conditions (2.30) to the δ term (2.28), another conservation equation is obtained, termed the 'Pythagoras delta equation':

$$0 = yQ + zR - xP\text{, the Pythagoras delta equation.} \tag{2.38}$$

In fact, all five conservation equations (2.31), (2.32), (2.36), (2.37) and (2.38) are related to the inner products of the eigenvectors of the matrix $\mathbf{A}_{30}$, (2.39) below, and there is also a sixth conservation equation that completes the set; see Section (2.14).

Under Pythagoras conditions, the matrix $\mathbf{A}_3$ simplifies as follows, and is also relabelled $\mathbf{A}_{30}$, where the extra subscript of zero denotes it is subject to these conditions:

$$\mathbf{A}_{30} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix}. \tag{2.39}$$

In pursuit of the conservation equations as vector inner products, the eigenvectors are given next.

2.10 The Pythagorean Eigenvectors

Having already defined all the necessary variables, the Pythagorean eigenvectors are stated here without proof as follows, the reader is referred to [1] for full details.

$$\mathbf{X}_{3+} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \mathbf{X}_{30} = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix},\ \mathbf{X}_{3-} = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}, \tag{2.40}$$

$$\mathbf{A}_{30}\mathbf{X}_{3+} = C\mathbf{X}_{3+},\ \mathbf{A}\mathbf{X}_{30} = 0,\ \mathbf{A}_{30}\mathbf{X}_{3-} = -C\mathbf{X}_{3-}\,.$$

The reciprocal, row-eigenvectors $\mathbf{X}^{3+}$, $\mathbf{X}^{30}$ and $\mathbf{X}^{3-}$ are defined by the following eigenvector equations:

$$\mathbf{X}^{3+}\mathbf{A}_{30} = C\mathbf{X}^{3+},\ \mathbf{X}^{30}\mathbf{A}_{30} = 0,\ \mathbf{X}^{3-}\mathbf{A}_{30} = -C\mathbf{X}^{3-},$$

and obtained from their standard counterparts using the following '$\mathbf{T}$ operator' and relations

$$\mathbf{T}_3 = \mathbf{T}^3 = \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\text{, the URM3 } \mathbf{T} \text{ operator} \tag{2.41}$$

$$\mathbf{X}^{3+} = \left(\mathbf{T}^3\mathbf{X}_{3-}\right)^T,\ \mathbf{X}^{30} = \left(\mathbf{T}^3\mathbf{X}_{30}\right)^T,\ \mathbf{X}^{3-} = \left(\mathbf{T}^3\mathbf{X}_{3+}\right)^T, \tag{2.42}$$

which gives

$$\mathbf{X}^{3+} = \begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix},\ \mathbf{X}^{30} = \begin{pmatrix}P & -Q & -R\end{pmatrix},\ \mathbf{X}^{3-} = \begin{pmatrix}x & y & -z\end{pmatrix}. \tag{2.43}$$

The Minkowski-like metric form of the $\mathbf{T}$ operator (2.41) is no coincidence - see also $\mathbf{T}_5$ (4.7) and URMT's five-dimensional STR solution in Section (4).

2.11 Conjugate Forms

The reciprocal, row eigenvectors (2.43) are also known as conjugates, and denoted by an over-struck bar as in the following definitions:

$$\overline{\mathbf{X}}_{3-} = \mathbf{X}^{3+},\ \overline{\mathbf{X}}_{30} = \mathbf{X}^{30} \text{ and } \overline{\mathbf{X}}_{3+} = \mathbf{X}^{3-}, \text{ conjugate forms.} \tag{2.44}$$

Conjugate vectors, e.g. $\overline{\mathbf{X}}_{3-}$, contract with their standard forms, such as $\mathbf{X}_{3-}$ in this example, to give real scalars, $\overline{\mathbf{X}}_{3-}\mathbf{X}_{3-} = 0$ (2.54), further below. This contraction to give a scalar is in accordance with both the rules of matrix (vector) multiplication and tensor algebra.

2.12 The URM3 Parametric Solution

The URM3 eigenvector problem, under Pythagoras conditions (2.30), is a completely solved problem, with the coordinates x, y, z given by the standard parameterisation for two integers k and l

$$k, l \in \mathbb{Z},\ (k,l) \neq (0,0),\ \gcd(k,l) = 1, \tag{2.45}$$

$$x = 2kl,\ y = (l^2 - k^2),\ z = (l^2 + k^2). \tag{2.46}$$

The dynamical variables P, Q, R and scale factors α, β, γ are obtained in terms of both integers k and l, and two new integers s and t, which are actually solutions to the following Linear Diophantine equation (LDE):

$$+C = ks - lt,\ s, t \in \mathbb{Z}. \tag{2.47}$$

This LDE always has a solution since $\gcd(k,l) = 1$ (2.45), and once a particular solution s' and t' is obtained, by algorithmic means [5], then an infinite family of solutions can be generated, denoted here by integers s and t, and parameterised by another arbitrary integer t_3 as follows:

$$s = s' + t_3 l,\ t = t' + t_3 k,$$
$$t_3 \in \mathbb{Z},\ s', t' \in \mathbb{Z},\ (s', t', t_3) \neq (0,0,0). \tag{2.48}$$

With a solution for s and t obtained, then the dynamical variables and scale factors are given by

$$P = -(ks + lt),\ Q = (ls - kt),\ R = -(ls + kt), \tag{2.49}$$

$$\alpha = -2st,\ \beta = (t^2 - s^2),\ \gamma = (t^2 + s^2). \tag{2.50}$$

2.13 Eigenvector Temporal Evolution

By virtue of the arbitrary parameter t_3 (2.48) in the previous section, two of the three eigenvectors, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ evolve with t_3, which can be identified as a temporal parameter under the SPI, Section (2.15). In fact, this parameter t_3 is also identical to the negative of the global variation parameter δ in (2.27), i.e. $t_3 = -\delta$. By substituting the parametric solution for P, Q, R (2.49) and α, β, γ (2.50) into the eigenvectors (2.40), the following eigenvector evolution equations can be obtained; see also [1] (using symbol t_3 instead of m), where the initial values at time zero, $t_3 = 0$, are superscripted with a prime, i.e.

$$\mathbf{X}'_{3+} = \mathbf{X}_{3+}(t_3 = 0),\ \mathbf{X}'_{30} = \mathbf{X}_{30}(t_3 = 0),\ \mathbf{X}'_{3-} = \mathbf{X}_{3-}(t_3 = 0),\ t_3 \sim m \sim -\delta \text{ in [1].} \quad (2.51)$$

The eigenvector evolution equations in standard vector form are:

$$\mathbf{X}_{3+} = \mathbf{X}'_{3+}, \text{ static, no } t_3 \text{ dependence,}$$
$$\mathbf{X}_{30}(t_3) = -t_3\mathbf{X}_{3+} + \mathbf{X}'_{30},$$
$$\mathbf{X}_{3-}(t_3) = -t_3^2\mathbf{X}_{3+} + 2t_3\mathbf{X}'_{30} + \mathbf{X}'_{3-}, \quad (2.52)$$

and their reciprocal forms, using (2.42), are

$$\mathbf{X}^{3-} = \mathbf{X}'^{3-}, \text{ static, no } t_3 \text{ dependence,}$$
$$\mathbf{X}^{30}(t_3) = -t_3\mathbf{X}^{3-} + \mathbf{X}'^{30},$$
$$\mathbf{X}^{3+}(t_3) = -t_3^2\mathbf{X}^{3-} + 2t_3\mathbf{X}'^{30} + \mathbf{X}'^{3+},$$

From these evolution equations, the calculus relations given in Section (2.16) can be verified.

The reader is referred to [1] for a geometric account of URM3 eigenvector evolution.

2.14 Conservation Equations and Invariants

The six key conservation equations of URMT (the first five have already been given earlier), as obtained from the inner product relations between the eigenvectors and their reciprocals (or conjugates (2.44)), are given here for URM3. See Appendix (F) in either [2] or [3] for the general, n-dimensional case. The 5-dimensional variants, as required for STR, are given further in this paper. Note that the complete set of conservation equations is only valid under URMT Pythagoras conditions, e.g. (2.30) for URM3:

$$\mathbf{X}^{3-}\mathbf{X}_{3+} = \overline{\mathbf{X}}_{3+}\mathbf{X}_{3+} = x^2 + y^2 - z^2 = 0 \text{ Pythagoras (zero norm),} \quad (2.53)$$

$$\mathbf{X}^{3+}\mathbf{X}_{3-} = \overline{\mathbf{X}}_{3-}\mathbf{X}_{3-} = \alpha^2 + \beta^2 - \gamma^2 = 0 \text{ Pythagoras (zero norm),} \quad (2.54)$$

$$\mathbf{X}^{30}\mathbf{X}_{30} = \overline{\mathbf{X}}_{30}\mathbf{X}_{30} = P^2 + Q^2 - R^2 = +C^2 \text{ the DCE,} \quad (2.55)$$

$$\mathbf{X}^{3+}\mathbf{X}_{3+} = \mathbf{X}^{3-}\mathbf{X}_{3-} = \alpha x + \beta y + \gamma z = +2C^2, \text{ the potential equation, } V = 0 \text{ (2.34),}$$

$$\mathbf{X}^{30}\mathbf{X}_{3+} = \mathbf{X}^{3-}\mathbf{X}_{30} = xP - yQ - zR = 0\text{, the delta equation,}$$

$$\mathbf{X}^{30}\mathbf{X}_{3-} = \mathbf{X}^{3+}\mathbf{X}_{30} = \alpha P - \beta Q + \gamma R = 0\text{, the dual delta equation.}$$

The first two, Pythagoras equations are equivalent to the norms of $\mathbf{X}_{3+}$ and $\mathbf{X}_{3-}$, where the norm $\|\mathbf{X}\|$ of a vector $\mathbf{X}$ is defined in URMT as the inner product of the column vector $\mathbf{X}$ with its conjugate (or reciprocal) row vector $\overline{\mathbf{X}}$, i.e. using relations (2.44) then the norms are zero as in

$$\|\mathbf{X}_{3+}\|^2 = \overline{\mathbf{X}}_{3+}\mathbf{X}_{3+} = \mathbf{X}^{3-}\mathbf{X}_{3+} = 0\text{, by (2.53),} \tag{2.56}$$

$$\|\mathbf{X}_{3-}\|^2 = \overline{\mathbf{X}}_{3-}\mathbf{X}_{3-} = \mathbf{X}^{3+}\mathbf{X}_{3-} = 0\text{, by (2.54).}$$

2.15 The Standard Physical Interpretation

The URMT standard physical interpretation (SPI) of all variables, eigenvectors and matrices, as first given for URM3 in [1], and Appendix (J) in [2] and [3], is reproduced below.

Following the publication of [3], this is not the only physical interpretation and, indeed, the URMT Harmonic Oscillator in [3] is a 'dual', physical interpretation. Nevertheless, this SPI seems the best as regards the STR solution, which is detailed in the next few sections.

$\mathbf{X}_{3+}$, $\mathbf{X}^{3-}$, $\mathbf{\Delta}$, x, y, z, acceleration or force per unit mass, LT^{-2}

$\mathbf{A}_3$, $\mathbf{A}_{30}$, $\mathbf{X}_{30}$, $\mathbf{X}^{30}$, P, Q, R, $\overline{P}, \overline{Q}, \overline{R}$, C, velocity or momentum per unit mass, LT^{-1}

$\mathbf{X}^{3+}$, $\mathbf{X}_{3-}$, α, β, γ, position, L

t_3, m, δ, time, T

K, V, C^2, velocity squared or total energy E ($E = C^2$) per unit mass, L^2T^{-2}.

Note that all conjugate quantities have the same physical units as their standard forms, e.g. $units(\mathbf{X}_{3+}) = units(\overline{\mathbf{X}}_{3+}) = units(\mathbf{X}^{-})$ = acceleration = LT^{-2}. This is not the case for the related reciprocal quantities, e.g. $units(\mathbf{X}_{3+}) \neq units(\mathbf{X}^{3+})$ and $units(\mathbf{X}_{3-}) \neq units(\mathbf{X}^{3-})$, except when they are self-reciprocal, i.e. in the case of the zero eigenvectors $\mathbf{X}_{30}$ and $\mathbf{X}^{30}$, where $units(\mathbf{X}_{30}) = units(\mathbf{X}^{30})$. The physical units of all conjugate and reciprocal vectors can be determined from the $\mathbf{T}$ operator relations (2.42) using the above units for the standard forms.

2.16 URM3 Calculus Relations

Whilst URM3 vectors $\mathbf{X}_{3+}$, $\mathbf{X}_{30}$, $\mathbf{X}_{3-}$ can be consistently interpreted in terms of their physical units, with an acceleration, velocity and position vector respectively, they are also related via the following calculus relations, further justifying the standard physical interpretation as given above. Keep in mind there is no explicit calculus, i.e. limiting process, in URMT.

The standard calculus derivative $\frac{d}{dt_3}$ ($\sim\frac{d}{dm}$ in [1]) is used as a good, large t_3 approximation for discrete differences Δ, i.e.

$$\frac{d}{dt_3} \approx \frac{\delta}{\delta t_3} \sim \boldsymbol{\Delta} t_3,\ t_3 >> 0,\ \Delta t_3 = \delta t_3 = 1,$$

$$\frac{d\mathbf{X}_{3-}}{dt_3} = 2\mathbf{X}_{30}, \text{ derivative of position = twice velocity } (\mathbf{X}_{30}),$$

$$\frac{d\mathbf{X}_{30}}{dt_3} = -\mathbf{X}_{+}, \text{ derivative of velocity = acceleration } (-\mathbf{X}_{3+}),$$

$$\frac{d\mathbf{X}_{3+}}{dt_3} = 0, \text{ derivative of acceleration = zero (constant acceleration)},$$

$$\frac{d^2\mathbf{X}_{3-}}{d^2 t_3} = -2\mathbf{X}_{3+}, \text{ second derivative of position = twice acceleration } (-\mathbf{X}_{3+}).$$

2.17 Geometric and Physical Aspects

So far, all URMT's properties have been algebraically expressed, but the eigenvector solution also possesses some interesting geometric properties, as now described.

The two eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$, for the two, non-zero eigenvalues $\pm C$, are Pythagorean, i.e. they satisfy the Pythagoras equation (4.29) and (4.30), and have zero norm (4.49). Because they satisfy Pythagoras, they each form a 2D, discrete cone in 3D, ultimately parameterised by three integers k, l (4.50) and t_3 (4.53), where the third parameter t_3 is temporal. The set of all points covered by these parameters represents an infinite set of eigenvectors, and is denoted by the two cone sets $\mathbf{C}_U$ and $\mathbf{C}_L$ for $\mathbf{X}_+$ and $\mathbf{X}_-$ respectively. For each point in $\mathbf{C}_U$, i.e. fixed $\mathbf{X}_+$, the position eigenvector $\mathbf{X}_-$ evolves with time t_3. For large t_3, see (4.58), it changes by multiples of $\mathbf{X}_+$ and, given both it and $\mathbf{X}_+$ are Pythagorean, with a zero norm (4.49), it effectively traces a null trajectory in the cone $\mathbf{C}_L$. Furthermore, this trajectory has inverse square law curvature with respect to time t_3 [1]#3, Since it is also at a zero, constant potential at every point (4.32), there are no forces acting in the direction of motion and it therefore possesses a constant kinetic energy (per unit mass) of C^2. It is thus physically interpreted as the null (zero norm), geodesic trajectory of a massless particle (with C equated to the speed of light *c*) in free-fall.

Because $\mathbf{X}_+$ and $\mathbf{X}_-$ can never be zero, due to the non-zero value of eigenvalue C (4.0), the cones $\mathbf{C}_U$ and $\mathbf{C}_L$ actually have no tip, i.e. there is no point (0,0,0), and this is termed 'no-singularity' in URMT for obvious reasons.

As regards the zero eigenvector $\mathbf{X}_0$, it represents a velocity (4.42), and its solution space forms a 2D, discrete hyperbolic sheet in 3D, denoted by the infinite set of points $\mathbf{H}$ [1]. The discrete hyperbolic sheet is the DCE, i.e. the conservation equation (4.45) in $\mathbf{X}_0$, where the elements of $\mathbf{X}_0$ are the dynamical variables P, Q, R. Like $\mathbf{X}_-$, $\mathbf{X}_0$ also evolves with time t_3.

Taken together, the union of the sets $\mathbf{C}_L$, $\mathbf{C}_U$ and $\mathbf{H}$ forms the discrete lattice $\mathbf{L}$, which represents the complete URM3 eigenvector solution. As time t_3 progresses, the discrete hyperbolic sheet of $\mathbf{X}_0$ converges (asymptotically) on to the cone $\mathbf{C}_L$, both of which align anti-parallel to $\mathbf{C}_U$, and the solution is said to 'flatten' [1]#3. At every point in the lattice, the conservation equations, Section (**Error! Reference source not found.**), i.e. inner products between the eigenvectors, give the same set of invariants, $\{0,\pm C^2,\pm 2C^2\}$, and for unity eigenvalue this is just $\{0,\pm 1,\pm 2\}$, i.e. the most basic units possible. Note that the negative values can be achieved by reversing the sign of the $\mathbf{T}$ operator (4.39) without detriment to the eigenvectors (4.41). These integer invariants hint at a fundamental quantisation of conserved quantities such as charge, spin etc.

This completes the review of the 3x3 fundamentals of URMT, the paper now proceeds to higher-dimensional extensions and their application to STR.

3 STR Higher Dimensional Extensions

Under the SPI, Section (2.15), which includes URM3 Pythagoras conditions as standard, and hence quadratic, Diophantine conservation equations, Section (2.14), URM3 is effectively limited to a two-dimensional subspace of three-spatial dimensions, i.e. the discrete cone surface that is represented by the zero norm eigenvectors, $\mathbf{X}_{3+}$ and $\mathbf{X}_{3-}$, whose elements satisfy the Pythagoras equation (2.53) and (2.54) respectively. There is also the discrete hyperbolic surface, eigenvector $\mathbf{X}_{30}$, that represents the DCE (2.55). Including a time-dimension t_3 (2.48) as the variational (evolutionary) parameter, the URM3 solution is physically considered to be a two-spatial, one-time ('2+1') dimensional representation of a massless particle moving at the speed of light. For particles with mass, sub-luminal speeds, and a consequent non-zero relativistic interval $c\tau$, (4.1b) further below, five dimensions are actually required in URMT (three spatial x, y, z, one each for laboratory time t and proper time τ), hence the need to extend URM3 to higher dimensions (notably five here), as first published in [2].

The extension to higher dimensions n, where $n > 3$ (n is not the exponent (2.5) here), naturally involves extending to $n \times n$ unity root matrices and n-element eigenvectors, whilst retaining an eigenvector solution with the same URM3 invariant, zero potential , Section (2.9), as a starting point, to give compatibility with URM3. Doing this ensures that the same two, non-zero eigenvalues appear ($\pm C$), with all others zero, accompanied by the same set of Pythagorean, eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$, but now with additional zero eigenvectors $\mathbf{X}_{0A,B,C,}$ etc., one for each repeated zero eigenvalue. The resulting solution shows that each excess dimensional element of $\mathbf{X}_0$ (four and higher) comprises a single, non-zero term in the eigenvalue C (3.13b), later equated with the speed of light (4.13c).

3.1 The URM5 Unity Root Matrix

To extend the number of spatial dimensions to a full three, as in the world around us, plus add one time dimension, and also an extra dimension for the proper time, the URM3 unity root matrix (2.8) is extended to a 5x5 matrix $\mathbf{A}_5$ as follows:

$$\mathbf{A}_5 = \begin{pmatrix} 0 & M & H & N & J \\ \overline{M} & 0 & S & T & U \\ \overline{H} & \overline{S} & 0 & R & \overline{Q} \\ \overline{N} & \overline{T} & \overline{R} & 0 & P \\ \overline{J} & \overline{U} & Q & \overline{P} & 0 \end{pmatrix}, \tag{3.1}$$

comprising ten dynamical variables

$$M, H, N, J \in \mathbb{Z}, \text{(URM5)}, \; S, T, U \in \mathbb{Z}\,, \text{(URM4)}, \tag{3.2}$$
$$P, Q, R \in \mathbb{Z}, \; (P, Q, R) \neq (0,0,0) \; \text{(2.5), (URM3)},$$

and their conjugates

$$\overline{M}, \overline{H}, \overline{N}, \overline{J} \in \mathbb{Z}, \text{(URM5)}, \; \overline{S}, \overline{T}, \overline{U} \in \mathbb{Z}, \text{(URM4)}, \tag{3.3}$$

$$\overline{P},\overline{Q},\overline{R} \in \mathbb{Z},\ (\overline{P},\overline{Q},\overline{R}) \neq (0,0,0)\ \text{(2.5), (URM3).}$$

The lower-right, 3x3 sub-matrix of $\mathbf{A}_5$ (3.1) is the same as the URM3 matrix $\mathbf{A}_{30}$ (2.39).

Only the URM3 dynamical variables P,Q,R and $\overline{P},\overline{Q},\overline{R}$ are true integer, unity roots (2.5), whilst the new dynamical variables M,H,N,J (URM5) and S,T,U (URM4), plus their conjugates $\overline{M},\overline{H},\overline{N},\overline{J}$ and $\overline{S},\overline{T},\overline{U}$ respectively, are defined by the URM5 Pythagoras conditions, (3.4) below, but are not unity roots. Neither are the additional dynamical variables necessarily restricted to the non-zero conditions, (3.2) and (3.3) above, albeit if they are all simultaneously zero then the theory is reduced back to URM3 since $\mathbf{A}_5$ is then just a zero-padded form of $\mathbf{A}_{30}$.

Sandwiched in between URM3 and URM5 is URM4, which is almost identical to URM5, barring its restriction to four dimensions. As a consequence, it is considered a three-spatial, one time dimension (3+1) STR solution for a zero relativistic interval, massless particle, and much the same as URM3 therefore. Albeit URM4 can have a non-zero potential, even when not under its own Pythagoras conditions (3.4), and the possible links with mass are discussed in Section (5) and [3].

3.2 URM5 Pythagoras Conditions

The above URM5 formulation is immediately simplified with the goal of producing analytic solutions with the same physical attributes, i.e. an invariant, zero potential as per URM3, where the elements of every eigenvector $\mathbf{X}$, for a non-zero eigenvalue ($\pm C$), obey the n-dimensional Pythagoras equation, and therefore have zero norm, e.g. (2.56). These simplifications start with the URM5 Pythagoras conditions, defined in an almost similar way to URM3 (2.30), i.e.

$$\begin{gathered}\overline{M} = -M\,,\ \overline{H} = -H\,,\ \overline{N} = -N\,,\ \overline{J} = J\,,\ \text{(URM5)},\\ \overline{S} = -S\,,\ \overline{T} = -T\,,\ \overline{U} = U\,,\ \text{(URM4)},\\ \overline{P} = P\,,\ \overline{Q} = Q\,,\ \overline{R} = -R\ \text{(URM3)}.\end{gathered} \tag{3.4}$$

Under these conditions, the matrix $\mathbf{A}_5$ (3.1) becomes $\mathbf{A}_{50}$:

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & M & H & N & J \\ -M & 0 & S & T & U \\ -H & -S & 0 & R & Q \\ -N & -T & -R & 0 & P \\ J & U & Q & P & 0 \end{pmatrix}. \tag{3.5}$$

The URM5 kinetic term K and potential term V are defined as follows:

$$K = J^2 + P^2 + Q^2 + U^2 - (H^2 + M^2 + N^2 + R^2 + S^2 + T^2)\,, \tag{3.6}$$

$$\begin{aligned} V = {} & [QT - (PS + RU)]^2 + [NQ - (JR + HP)]^2 \\ & + [HU - (JS + MQ)]^2 + [NU - (JT + MP)]^2 \\ & - [HT - (MR + NS)]^2 \end{aligned}\ . \tag{3.7}$$

and using these two terms the characteristic equation for matrix $\mathbf{A}_{50}$, eigenvalue λ, is

$$0 = \lambda(-\lambda^4 + K\lambda^2 + V), \tag{3.8}$$

If K and V can be reduced to their URM3 forms, i.e. $K = P^2 + Q^2 - R^2$ (2.33) and $V = 0$ (2.34), then the potential energy remains zero and the DCE (2.15) is just the constant kinetic energy:

$$C^2 = K\,. \tag{3.9}$$

The characteristic equation (3.8) will also simplify to

$$0 = \lambda^3(K - \lambda^2)\,,$$

and thus, using K (3.9), there will be two non-zero and three zero eigenvalues, i.e.

$$\lambda = \pm C,0,0,0\,. \tag{3.10}$$

So, to achieve the same results as the URM3, with the same SPI Section (2.15), the first goal is to obtain a constant kinetic energy and zero potential energy.

3.3 An Invariant Zero Potential

To obtain a constant kinetic energy (3.9) and zero potential (2.34), the matrix $\mathbf{A}_{50}$ (3.5) is written in the following block matrix form in terms of URM3 vectors, $\mathbf{X}_{3+}$ (2.40) and $\mathbf{X}^{3-}$ (2.43), unity root matrix $\mathbf{A}_{30}$ (2.39) and temporal evolutionary parameters t_4 and t_5

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & 0 & -t_5\mathbf{X}^{3-} \\ 0 & 0 & -t_4\mathbf{X}^{3-} \\ t_5\mathbf{X}_{3+} & t_4\mathbf{X}_{3+} & \mathbf{A}_{30} \end{pmatrix},\ t_4,\ t_5 \in \mathbb{Z}\,,\ units(t_4, t_5) = T\,,\ \text{time} \tag{3.11}$$

Expanding $\mathbf{A}_{50}$ in full gives

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & 0 & -t_5x & -t_5y & +t_5z \\ 0 & 0 & -t_4x & -t_4y & +t_4sz \\ +t_5x & +t_4x & 0 & R & Q \\ +t_5y & +t_4y & -R & 0 & P \\ +t_5z & +t_4z & Q & P & 0 \end{pmatrix},$$

and by comparison with $\mathbf{A}_{50}$ (3.5), the dynamical variables, S,T,U and M,H,N,J are

$$M = 0\,,\ H = -t_5x\,,\ N = -t_5y\,,\ J = +t_5z\,, \tag{3.12}$$
$$S = -t_4x\,,\ T = -t_4y\,,\ U = +t_4z\,.$$

These conditions satisfy the URM5 Pythagoras conditions (3.4), but this does not guarantee an invariant, zero potential, unlike URM3. However, the reader can verify this particular solution does have zero potential energy by substituting for the dynamical variables from (3.12) into (3.7), and using Pythagoras (2.31). Likewise, the kinetic energy K (3.6) can be verified as reducing to C^2 (3.9).

The temporal parameters t_4 and t_5 are the evolutionary times in the fourth and fifth dimensions respectively, i.e. the first two rows and columns of $\mathbf{A}_{50}$, and complement URM3's temporal parameter t_3 (2.48) in the third dimension. They are later assigned to scaled forms of the familiar laboratory time t and proper time τ as part of the URMT relativistic Doppler solution, Section (4).

3.4 The URM5 Eigenvector Solution

Just like the URM3 solution under Pythagoras conditions, this URM5 solution presented here is also a completely solved problem with an analytic solution. In [2], Appendix (I16), it is called a 'lifted' solution since it is basically a lift of (or derived from) the URM4 solution, which itself is a lift of the URM3 solution, and thus always retains an invariant, zero potential, under Pythagoras conditions, by definition. The complete solution set of eigenvectors to $\mathbf{A}_{50}$ (3.5) is given in block matrix form in terms of the URM3 eigenvector solution as follows:

$$\mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix},\ \lambda = C,\ \mathbf{A}_{50}\mathbf{X}_{5+} = C\mathbf{X}_{5+}, \tag{3.13a}$$

$$\mathbf{X}_{5-} = -(t_5^2 + t_4^2)\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + 2t_5\begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + 2t_4\begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3-} \end{pmatrix}, \tag{3.13b}$$

$$\lambda = -C,\ \mathbf{A}_{50}\mathbf{X}_{5-} = -C\mathbf{X}_{5-}$$

$$\mathbf{X}_{50A} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{30} \end{pmatrix},\ \lambda = 0,\ \mathbf{A}_{50}\mathbf{X}_{50A} = 0, \tag{3.13c}$$

$$\mathbf{X}_{50B} = -t_4\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix},\ \lambda = 0,\ \mathbf{A}_{50}\mathbf{X}_{50B} = 0, \tag{3.13d}$$

$$\mathbf{X}_{50C} = -t_5\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix},\ \lambda = 0,\ \mathbf{A}_{50}\mathbf{X}_{50C} = 0. \tag{3.13e}$$

The eigenvector equations are as per URM3 (2.40), but now with three zero eigenvectors $\mathbf{X}_{50A}$, $\mathbf{X}_{50B}$ and $\mathbf{X}_{50C}$ instead of just one, $\mathbf{X}_{30}$. This solution (3.13) forms the basis of the URMT formulation of STR as studied in the remainder of this paper.

Whilst the eigenvectors $\mathbf{X}_{5+}$ and $\mathbf{X}_{50A}$ are just embedded forms of their three-dimensional, URM3 equivalents $\mathbf{X}_{3+}$ and $\mathbf{X}_{30}$, with a zero fourth and fifth dimensional contribution (zero first two elements), the remaining three, $\mathbf{X}_{5-}$, $\mathbf{X}_{50B}$ and $\mathbf{X}_{50C}$ are not so simple. In particular, $\mathbf{X}_{5-}$ is a full five-element position vector and will be used to represent an STR event, (4.1) further below. Of the other two zero eigenvectors, $\mathbf{X}_{50B}$ is the URM4, 4-dimensional equivalent of URM3's $\mathbf{X}_{30}$, and $\mathbf{X}_{50C}$ is the URM5 equivalent.

Keep in mind that the eigenvalue C is a velocity quantity, as per all dynamical variables under the SPI, and will be equated with the speed of light c, (4.13c) further below.

The URM5 SPI is identical to that of URM3 (and URM4, not detailed here), Section (2.15). It has all the same physical properties as per URM3, but is now extended to five dimensions, this includes the natural calculus properties amongst the eigenvectors; Section (3.5) below. Lastly, note that the $\mathbf{X}_{5-}$ solution (3.13b) is quadratic in the evolutionary parameters t_4 and t_5 for the URM3 $\mathbf{X}_{3+}$ component only, i.e. the last three elements, whilst the first two elements, $2Ct_5$ and $2Ct_4$ respectively, are linear in t_5 and t_4. This means that, for large evolutionary times $t_4, t_5 >> 0$, the first two elements (fifth and fourth dimensions respectively) shrink relative to the first three dimensions (the last three elements), the solution thus exhibits the geometric property of compactification, i.e. the apparent shrinkage of higher dimensions with respect to the lower dimensions, which is fully detailed in [2].

3.5 URM5 Calculus Relations

The URM3 calculus relations, Section (2.16), are also satisfied for this URM5 lifted solution (3.13). With more than one evolutionary parameter for four and higher dimensions, the standard calculus partial derivative $\frac{\partial}{\partial t_i}$ is now used in place of $\frac{d}{dt_3}$ for derivatives with respect to evolutionary time t_i, $i = 4,5$ here, i.e.

$$\frac{\partial}{\partial t_i} \approx \frac{\delta}{\delta t_i} \sim \mathbf{\Delta} t_i,\ t_i >> 0,\ \mathbf{\Delta} t_i = \delta t_i = 1.$$

For URM5, the partial derivatives in the fourth and fifth dimensions are as follows:

$\frac{\partial \mathbf{X}_{5-}}{\partial t_4} = 2\mathbf{X}_{50B}$, derivative of position = twice velocity ($\mathbf{X}_{50B}$)

$\frac{\partial \mathbf{X}_{5-}}{\partial t_5} = 2\mathbf{X}_{50C}$, ditto ($\mathbf{X}_{50c}$)

$\frac{\partial \mathbf{X}_{50A}}{\partial t_3} = -\mathbf{X}_{5+}$, derivative of velocity = acceleration ($-\mathbf{X}_{5+}$)

$\frac{\partial \mathbf{X}_{50B}}{\partial t_4} = -\mathbf{X}_{5+}$,

$\dfrac{\partial \mathbf{X}_{50C}}{\partial t_5} = -\mathbf{X}_{5+}$, ditto

$\dfrac{\partial \mathbf{X}_{5+}}{\partial t_4} = 0$, $\dfrac{\partial \mathbf{X}_{5+}}{\partial t_5} = 0$, constant acceleration

$\dfrac{\partial^2 \mathbf{X}_{5-}}{\partial^2 t_4} = -2\mathbf{X}_{5+}$, $\dfrac{\partial^2 \mathbf{X}_{5-}}{\partial^2 t_5} = -2\mathbf{X}_{5+}$ 2nd derivative of position = twice acceleration ($-\mathbf{X}_{5+}$)

4 The STR Doppler Solution

This section maps the URM5 lifted solution, given in the previous section, to an STR event with a non-zero, relativistic interval, with the goal of representing a non-zero rest mass particle travelling at a sub-luminal velocity.

4.1 An STR Event as an Eigenvector

Under the SPI, the URM5 minus eigenvector, i.e. $\mathbf{X}_{5-}$ (3.13b), represents a position vector as for URM3 $\mathbf{X}_{3-}$, Section (2.15), and therefore this five-element vector is suitable to represent what is, in STR, a four-vector position (or event). The vector $\mathbf{X}_{5-}$ (3.13b) is expanded in full, below, with a side-by-side definition of a five-vector form of an STR event given in terms of position coordinates x, y, z (see the following note), proper time τ, laboratory time t and speed of light c:

$$\mathbf{X}_{5-} = \begin{pmatrix} 2t_5C \\ 2t_4C \\ \mathbf{X}_{3-} - (t_5^2 + t_4^2)\mathbf{X}_{3+} \end{pmatrix} \text{URM5, (4.1a), } \mathbf{X}_{5-} = \begin{pmatrix} -c\tau \\ x \\ y \\ z \\ -ct \end{pmatrix} \text{STR (4.1b).} \tag{4.1}$$

Important. The variables x, y, z here have reverted to their more familiar spatial definition, i.e. as position coordinates, rather than their URM3 interpretation as accelerations - the role of spatial coordinates is usually played by α, β, γ in the SPI, Section (2.15). This switch to using x, y, z is for notational convenience (familiarity) only, and the eigenvectors $\mathbf{X}_{3+}$ and $\mathbf{X}_{5+}$ remain as acceleration vectors under the SPI, with eigenvectors $\mathbf{X}_{3-}$ and $\mathbf{X}_{5-}$ remaining as position vectors.

The reason to use a minus sign in the first and last elements of $\mathbf{X}_{5-}$ (4.1b) is purely one of URMT convention, and done to keep the last element same as the last element '$-\gamma$' of URM3's position eigenvector $\mathbf{X}_{3-}$ (2.40), which then also dictates the sign of the first element of $\mathbf{X}_{5-}$.

Both $\mathbf{X}_{5-}$ vectors in (4.1) have a zero norm (see (4.5) below) and, in particular, the STR five-vector norm is zero in accordance with an invariant, non-zero, STR interval $c\tau$, i.e.

$$(c\tau)^2 + x^2 + y^2 + z^2 - (ct)^2 = 0, \tag{4.2}$$

where τ and t are related by the usual STR definitions:

$$t = \gamma\tau, \tag{4.3}$$

$$\gamma = \frac{1}{\sqrt{\left(1 - (v/c)^2\right)}}, \ \gamma \geq 1, \ v \leq c. \tag{4.4}$$

Note that this γ is not the same as the scale factor γ in (2.7).

The zero norm (4.2) is represented in URMT by a conservation equation, i.e. the inner product of $\mathbf{X}_{5-}$ and its reciprocal eigenvector $\mathbf{X}^{5+}$ (derived further below), i.e.

$$\|\mathbf{X}_{5-}\|^2 = \mathbf{X}^{5+}\mathbf{X}_{5-} = 0, \tag{4.5}$$

which is basically the 5D Pythagoras equation; see (2.56) for the URM3, zero norm cases.

Note that what is traditionally a non-zero, four-vector interval $c\tau$ in STR, is converted to a five-vector, zero norm in URMT. URMT treats numerous physical problems by adding an extra dimension to the eigenvector and converting it to a zero norm, Pythagorean eigenvector, see [3].

The reciprocal, row eigenvector $\mathbf{X}^{5+}$ (or conjugate $\overline{\mathbf{X}}_{5-}$) is defined in the usual URMT way by

$$\mathbf{X}^{5+} = \overline{\mathbf{X}}_{5-} = (\mathbf{T}^5\mathbf{X}_{5-})^T, \tag{4.6}$$

where the 5×5 URM5 matrix operator $\mathbf{T}_5$ $(=\mathbf{T}^5)$ is defined in block matrix form using the 4×4 identity matrix $\mathbf{I}_4$ as follows

$$\mathbf{T}_5 = \mathbf{T}^5 = \begin{pmatrix} \mathbf{I}_4 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.7}$$

so that $\mathbf{X}^{5+}$ becomes, according to (4.6),

$$\mathbf{X}^{5+} = \begin{pmatrix} -c\tau & x & y & z & ct \end{pmatrix}.$$

It is noted that $\mathbf{T}_5$ is a 5D form of the more familiar 4x4 Minkowski metric [7].

The URM5 $\mathbf{X}_{5-}$ eigenvector (4.1a) is parameterised by the two, evolutionary, temporal parameters t_4 and t_5, and also the URM3 temporal parameter t_3, in accordance with the URM3 solution for $\mathbf{X}_{3-}$ (2.52), plus two of the four, non-temporal parameters k, l, s and t, Section (2.12), only two of which are independent.

Looking at the URM3 parametric solution in Section (2.12) and the URM5 eigenvector solution for $\mathbf{X}_{5-}$ (4.1a), the parameters (except t_5) are set to the following values to obtain the URMT 'Doppler solution' [3]:

$$k = 0,\ s = 0,\ t_3 = 0,\ u' = t,\ t_4 = 0, \tag{4.8}$$

Note that parameter t in (2.47) is replaced by u' here to avoid a clash of notation with the STR laboratory time t, where the prime on u also denotes an initial value.

With $t_3 = 0$, the $\mathbf{X}_{3-}$ vector is equal to its initial value, i.e. $\mathbf{X}_{3-}(t_3 = 0) = \mathbf{X}'_{3-}$ (2.51), which is given in terms of u' as follows, in accordance with (2.40), using $s = 0$, $u' = t$ (4.8) and (2.50):

$$\mathbf{X}_{3-} = \begin{pmatrix} 0 \\ u'^2 \\ -u'^2 \end{pmatrix}.$$

The URM3 vector $\mathbf{X}_{3+}$ (2.40) is invariant, and with $k = 0$ (4.8), then by (2.46) $\mathbf{X}_{3+}$ becomes

$$\mathbf{X}_{3+} = \begin{pmatrix} 0 \\ l^2 \\ l^2 \end{pmatrix}.$$

Substituting for $\mathbf{X}_{3+}$ and $\mathbf{X}_{3-}$ into $\mathbf{X}_{5-}$ (4.1a), and expanding in full, using $t_3 = t_4 = 0$ (4.8), gives

$$\mathbf{X}_{5-}(t_5, u') = \begin{pmatrix} 2Ct_5 \\ 0 \\ 0 \\ -t_5^2 l^2 + u'^2 \\ -t_5^2 l^2 - u'^2 \end{pmatrix}. \tag{4.9}$$

The URMT position vector $\mathbf{X}_{5-}$ is thus seen to be a function of temporal parameters t_5, u' and l, where u' and l are related to each other by the LDE (2.47), with u' replacing t, and $s = 0$ (4.8), as in

$$l = -\frac{C}{u'}. \tag{4.10}$$

Given l is functionally dependent on u', then the URMT eigenvector $\mathbf{X}_{5-}$ (4.9) is parameterised by just the two parameters t_5 and u'.

The STR vector $\mathbf{X}_{5-}$ (4.1b) is also simplified to be compatible with the above, simplified form of $\mathbf{X}_{5-}$ (4.9), with motion (velocity v_z) along the z axis only, as follows:

$$\mathbf{X}_{5-} = \begin{pmatrix} -c\tau \\ 0 \\ 0 \\ z \\ -ct \end{pmatrix},\ x = 0,\ y = 0,\ z = v_z t,\ v = |v_z|. \tag{4.11}$$

4.2 The Doppler Parameterisation

With both the URMT and STR forms of the $\mathbf{X}_{5-}$ simplified, the problem now is to map the two URMT parameters t_5 and u' in $\mathbf{X}_{5-}$ (4.9) to the STR equivalent (4.11). By comparing these two vectors, the STR parameters ct and z (with τ derived from t by (4.3)) and URMT parameters t_5

and u' (with l derived from u' by (4.10)), are mapped to each other as follows, in particular, the eigenvalue C is finally equated with the speed of light c:

$$\tau = -2t_5, \tag{4.12a}$$

$$x = 0, \tag{4.12b}$$

$$y = 0, \tag{4.12c}$$

$$c = C, \tag{4.12d}$$

$$z = u'^2 - t_5^2 l^2,\ l = -\frac{c}{u'}, \tag{4.12e}$$

$$ct = u'^2 + t_5^2 l^2, \tag{4.12f}$$

$$v_z = c\left(\frac{u'^2 - t_5^2 l^2}{u'^2 + t_5^2 l^2}\right), \tag{4.12g}$$

Conversely, the URMT parameters t_5 and u' are given in terms of STR parameters ct and z by

$$t_5 = -\frac{\sqrt{(ct)^2 - z^2}}{2c}, \tag{4.13a}$$

$$u' = \sqrt{\frac{(ct + z)}{2}}, \tag{4.13b}$$

$$C = c, \tag{4.13c}$$

Some numeric examples of this parameterisation are given at the end of Section (6) in [3].

Notice from (4.12a) that URMT's temporal, variational parameter t_5 runs in the reverse direction to the proper time, and at half the rate. Whilst this could be changed by altering the eigenvalue relations (4.12d) or (4.13c), that equate C directly with c, this isn't necessary, and it is preferable to keep C and c identical.

From the form of z (4.12e) and v_z (4.12g), the following Doppler parameter α is formally defined in URMT as

$$\alpha = -\frac{u'^2}{ct_5}. \tag{4.14}$$

Note that this α is not the same as the scale factor α in (2.7).

The STR velocity ratio (or 'normalised velocity') β is obtained from v_z (4.12g) as follows, using $l = -c/u'$ (4.12e):

$$\beta = \frac{v_z}{c} = \left(\frac{u'^4 - (ct_5)^2}{u'^4 + (ct_5)^2}\right), \tag{4.15}$$

and can be expressed in terms of α^2, after some rearrangement, as the following dimensionless ratio

$$\alpha^2 = \left(\frac{1+\beta}{1-\beta}\right). \tag{4.16}$$

Using $\beta = v_z / c$ (4.15), this gives α as

$$\alpha = \sqrt{\left(\frac{c+v_z}{c-v_z}\right)},$$

which is none other than the dimensionless STR Doppler shift in wavelength [7].

Using α (4.14) and relations (4.12), the URM5 position eigenvector $\mathbf{X}_{5-}$ (4.9) can now be re-expressed as

$$\mathbf{X}_{5-} = \frac{ct}{(\alpha^2+1)}\begin{pmatrix} -2\alpha \\ 0 \\ 0 \\ +(\alpha^2-1) \\ -(\alpha^2+1) \end{pmatrix}.$$

The URM5 acceleration vector $\mathbf{X}_{5+}$ and 'velocity' vector $\mathbf{X}_{50C}$ are also reproduced below from [3] as follows:

$$\mathbf{X}_{5+} = \frac{c}{t}\left(\frac{\alpha^2+1}{\alpha^2}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{X}_{50C} = \frac{c}{\alpha}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

In fact, of most interest here is actually the above, invariant acceleration vector $\mathbf{X}_{5+}$, which is seen to be inversely proportional to the laboratory time t. With the two spatial coordinates x and y zero (second and third elements of $\mathbf{X}_{5+}$), the non-zero acceleration a_z is given by the fourth element, i.e.

$$a_z = \frac{c}{t}\left(\frac{\alpha^2+1}{\alpha^2}\right). \tag{4.17}$$

From this expression, for speeds close to the speed of light when $\beta \approx 1$ (4.15) and α is very large by (4.16), then the acceleration a_z is approximated as

$$a_z \approx \frac{c}{t}, \ \alpha >> 0,$$

which is just the Hubble-equivalent, expansion acceleration. Evidently, if $t = 0$ the solution blows up (quite literally) with a singularity.

Given the age of the universe is approximately 14 billion years, i.e. $O(10^{10})$ years, which equates to about 10^{17} seconds, and with $c \approx 3 \times 10^{8} m/s$ then the acceleration in this epoch is approximately

(4.68) $a_z \approx O(10^{-9})\ ms^{-2}$, URMT acceleration,

which is the acceleration equivalent of the Hubble constant $H = 73.8 \pm 2.4 \times 10^{3}\ ms^{-1}/Mpc$ (or $2.4 \times 10^{-18}\ ms^{-1}/m$), where $1Mpc = 3.086 \times 10^{22} m$. This derives as follows: at distance *r*, the recession velocity is given by the Hubble law as $v = Hr$, and differentiating this gives the acceleration $a = \dot{v} = H\dot{r}$. Using $v = \dot{r}$ this becomes $a = Hv$. For velocities near the speed of light $a = \dot{v} \approx Hc$ for $v \approx c$. Substituting for $H = 2.4 \times 10^{-18}\ ms^{-1}/m$ and $c = 3.0 \times 10^{8}\ ms^{-1}$ gives $a = 7.2 \times 10^{-10}\ ms^{-2}$.

The above gives the acceleration in the current epoch. However, at the very earliest, non-zero time, such as the Planck time $t = O(10^{-44})s$, then the acceleration is a huge $O(10^{50})ms^{-2}$.

5 The STR Mass Solution

This section extends the lifted form of the URM5 matrix $\mathbf{A}_{50}$ (3.11) by relaxing the constraint on dynamical variable M (3.12) such that it is now non-zero, and defined as the 'reduced velocity', detailed next.

5.1 The Reduced Velocity

In URMT, a very useful quantity, termed the 'reduced-velocity', and denoted by the symbol M, is defined as follows:

$$M = \frac{c}{\gamma},\ 0 \le M \le c \text{ for } \gamma \ge 1. \tag{5.1}$$

It is seen that with $\gamma > 1$ then M is less than the speed of light.

The following two limiting cases are of note, derived from the definition of γ (4.4),

$$v = c \ \Rightarrow \ \gamma = \infty,\ M = 0, \tag{5.2}$$

$$v = 0 \ \Rightarrow \ \gamma = 1,\ M = c. \tag{5.3}$$

For a massless particle, e.g. a photon or graviton with a velocity that of the speed of light, then case (5.2) applies and M is zero. For a particle at rest, with zero velocity, then case (5.3) applies and M is the speed of light c.

With these points in mind, M is termed 'reduced' because it is zero at the speed of light and grows to the speed of light as the speed v decreases to zero, i.e. it is the speed reduced from that of c, whereas v increases from 0 to c. When M is greater than zero it is considered equivalent to the speed v of a particle with finite mass and sub-luminal velocity, i.e. $v < c$ according to (5.4) below.

From M (5.1) and γ (4.4) the following important relation between c, M and v is obtained

$$0 = M^2 + v^2 - c^2, \tag{5.4}$$

which rearranges to give an alternative form for defining M as

$$M = \sqrt{c^2 - v^2},$$

where the positive root is taken for positive γ and c (5.1).

The relationship (5.4) is yet another Pythagoras equation, and there seems to be no escaping this simple equation throughout URMT physics.

What makes M extra special is that it is also a dynamical variable in the URM5 unity root matrix $\mathbf{A}_{50}$ (3.5), and not just an ad-hoc definition introduced for algebraic or physical convenience.

5.2 A Non-invariant, Non-zero Potential Solution

To obtain a non-zero, non-invariant potential energy term, the unity root matrix $\mathbf{A}_{50}$ (3.11) is now extended by addition of the reduced velocity M (5.1) to become

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & M & -t_5\mathbf{X}^{3-} \\ -M & 0 & -t_4\mathbf{X}^{3-} \\ t_5\mathbf{X}_{3+} & t_4\mathbf{X}_{3+} & \mathbf{A}_{30} \end{pmatrix}. \tag{5.5}$$

This M is the same dynamical variable M in the top row, second element of the general URM5 $\mathbf{A}_{50}$ matrix (3.5).

With a non-zero M, the URM5 characteristic equation (3.8) becomes

$$0 = \lambda(\lambda^2 - C^2)(\lambda^2 + M^2), \tag{5.6}$$

and the eigenvalues are thus

$$\lambda = \pm C, \pm iM, 0. \tag{5.7}$$

It is noted that if M is zero then the eigenvalues reduce to $\lambda = \pm C,0,0,0$ (3.10), as expected for a zero potential energy, (5.9) below.

In URM4 [2], two complex eigenvalues $\lambda = \pm iV$ also emerge for a non-zero potential V, and this suggests a connection between the URMT potential energy and mass, via the reduced, sub-luminal velocity M, as will become evident.

With a non-zero M, the kinetic term K (2.33) becomes

$$K = (P^2 + Q^2 - R^2) - M^2,$$

and keeping with $C^2 = P^2 + Q^2 - R^2$ from URM3 (TBD), the kinetic term is thus now

$$K = C^2 - M^2. \tag{5.8}$$

Additionally, the potential V is no longer zero but now

$$V = M^2. \tag{5.9}$$

Disregarding (or factoring) the eigenvalue λ for the zero eigenvalue in the characteristic equation (5.6), and rearranging, gives

$$\lambda^4 = (C^2 - M^2)\lambda^2 + M^2C^2. \tag{5.10}$$

Comparing this with the revised kinetic term (5.8) and potential (5.9), then it can also be written as

$$\lambda^4 = K\lambda^2 + VC^2 . \tag{5.11}$$

Lastly, for eigenvalues $\lambda = \pm C$ this characteristic equation becomes

$$C^4 = KC^2 + VC^2 , \tag{5.12}$$

and dividing throughout by C^2, which is always greater than zero by (2.4), returns the familiar DCE:

$$C^2 = K + V . \tag{5.13}$$

which justifies the revised kinetic and potential definitions, (5.8) and (5.9) respectively.

5.3 The Relativistic Energy-momentum Equation

Now to compare with STR. Throughout URMT, starting right at the beginning with URM3 in [1] and extending to URM4 and URM5 in [2], the characteristic equation, i.e. the DCE, is considered an energy conservation equation (per unit mass), with a total energy given by the invariant eigenvalue C^2. The eigenvalue C is inevitably associated with the speed of light c (4.13c), hence the familiar look to the relativistic energy formula $E = K = C^2 = c^2$ (per unit mass) for a zero potential, i.e. $V = 0$ in (2.17). Much of URMT, particularly in [2], concentrates on zero potential energy solutions and, as such, all the energy is kinetic. In essence, all very much like a particle with a zero rest mass but, nevertheless, finite energy C^2. This is why, repeatedly, the invariant, zero potential solution, Section (3.3), is considered primarily a massless particle solution, i.e. a photon or graviton. The introduction of a non-zero dynamical variable M now changes all that.

Returning to the characteristic equation (5.12), it is noted to be fourth order in C. Furthermore, given it splits nicely into two terms, i.e. a kinetic and potential term, it can be directly compared with the STR relativistic energy-momentum equation

$$E^2 = p^2c^2 + {E_0}^2 , \tag{5.14}$$

where, as usual, p is the momentum of an object (particle) with relativistic mass m and velocity v, i.e. $p = mv$, and the total energy E is given by Einstein's equation:

$$E = mc^2 , \tag{5.15}$$

with the rest mass (m_0) energy E_0 also:

$$E_0 = m_0c^2 . \tag{5.16}$$

Expanding (5.14) in component form, i.e.

$$(mc^2)^2 = (mv)^2c^2 + (m_0c^2)^2 , \tag{5.17}$$

and dividing throughout by the mass m^2 gives

$$c^4 = v^2 c^2 + \left(\frac{m_0}{m}\right)^2 c^4 . \tag{5.18}$$

Comparing this with the characteristic equation (5.12) gives the following associations of the energy terms, all, strictly speaking, per unit mass:

$$\begin{aligned} E &= C^2 = c^2 \text{, per unit mass,} \\ K &= C^2 - M^2 = v^2 \text{, ditto,} \\ V &= M^2 = \left(\frac{m_0}{m}\right)^2 c^2 \text{, ditto.} \end{aligned} \tag{5.19}$$

The kinetic term K is just the earlier definition (5.4) of the reduced velocity for $C = c$ (4.13c), i.e. $M^2 = C^2 - v^2$ (5.4). From STR [7], the ratio of masses m_0/m is equal to the reciprocal of γ (4.4), i.e.

$$\frac{1}{\gamma} = \frac{m_0}{m}, \tag{5.20}$$

and by substituting this ratio into the potential term (5.19), the dynamical variable M is seen to be related to the eigenvalue (now also the speed of light) as in its original definition $C = \gamma M$ (5.1).

5.4 The URMT Rest-mass Energy Equation

Lastly, and most importantly, using $C = \gamma M$ (5.1) and $C = c$ (4.13c), then the rest mass energy E_0 (5.16), becomes

$$E_0 = m_0 \gamma M C ,$$

and since $m = m_0 \gamma$ by (5.20) then the rest mass energy can be written in terms of the URMT dynamical variable M and eigenvalue C as

$$E_0 = mMC \text{, URMT's rest mass energy equation.} \tag{5.21}$$

Superficially, this is just a rewrite of $E_0 = m_0 c^2$, with the rest mass term $m_0 c$ effectively replaced by mM, but what was the STR set $\{m_0, m, v, c\}$, of two masses and two velocities, is now replaced by a single mass and three velocities, $\{m, M, v, C\}$, where the single, relativistic mass m now cancels across the energy-momentum equation (5.17) such that URMT's equivalent equation becomes the Pythagorean relation given earlier (5.4), rearranged and written in terms of the eigenvalue C as

$$C^2 = M^2 + v^2 \text{, URMT's energy-momentum equation.} \tag{5.22}$$

Whilst it may be argued that both (5.21) and (5.22) are just rewrite's of STR's equivalent energy equations, the reader is reminded that the URMT, invariant eigenvalue C originates from URM3 and a number-theoretic problem in linear algebra, Section (2.1). Furthermore, both energy equations, (5.21) and (5.22) are now symmetric, to within a sign, upon interchange of the velocities, i.e. M and C in $E_0 = mMC$, and M, v and C in $C^2 = M^2 + v^2$. In particular, this now puts the abstract, reduced velocity M on an equal footing with the familiar, terrestrial velocity v. The addition of M is seemingly just an extension of the URM5 lifted solution, yet it appears directly related to rest mass and also the potential energy (5.9).

It would seem, therefore, that by starting bottom-up with a problem in number theory, i.e. obtaining the integer eigenvalues and eigenvalues of the matrix $\mathbf{A}_{50}$ (5.5) according to an invariance principle, Section (2.6), the STR energy equations can be derived. Indeed, although not shown here, URMT can also relate a unity root matrix to both an event and its Lorentz transform [3].

6 STR to Newton II

So far, the URM5 STR solutions presented have been expressed in terms of the URM3 solution, whilst URM3 itself has not been explicitly expanded. However, as per Section (2.13), the URM3 eigenvector solution also evolves over time, and it is shown in this last section that, by equating URM3's evolutionary parameter t_3 to a scaled from of the proper time, the URM3 vector components can represent the position of an object under a constant acceleration as given by Newton's second law $F = ma$, or just simply 'Newton II'.

6.1 The Mass Eigenvector Solution

The unity root matrix $\mathbf{A}_{50}$ for the mass solution is reproduced below from (5.5)

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & M & -t_5\mathbf{X}^{3-} \\ -M & 0 & -t_4\mathbf{X}^{3-} \\ t_5\mathbf{X}_{3+} & t_4\mathbf{X}_{3+} & \mathbf{A}_{30} \end{pmatrix} \text{ (5.5)},$$

with eigenvalues

$$\lambda = \pm C, \pm iM, 0 \text{, (5.7)}.$$

Whilst M is now non-zero, denoting a sub-luminal velocity (5.1), and hence a particle with non-zero mass, the eigenvector $\mathbf{X}_{5+}$ remains the same embedded form of the invariant, URM3 acceleration eigenvector $\mathbf{X}_{3+}$, and likewise for the single, zero eigenvector $\mathbf{X}_{50}$, i.e.

$$\mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix}, \ \lambda = +C \text{ (3.13a)}, \quad \mathbf{X}_{50A} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{30} \end{pmatrix}, \ \lambda = 0 \text{ (3.13c)}.$$

The temporal parameter t_5 is related to the proper time τ as in the Doppler solution $\tau = -2t_5$ (4.12a), but the temporal parameter t_4 is not zeroed, unlike (4.8), and, instead, related to the laboratory time t as in

$$t_4 = -\frac{t}{2}. \tag{6.1}$$

With these settings, the five-vector position (STR event) eigenvector $\mathbf{X}_{5-}$ (4.1a) becomes

$$\mathbf{X}_{5-} = c^2 \frac{\left(1+\gamma^2\right)}{\gamma^2} \begin{pmatrix} 0 \\ -ct \\ \mathbf{X}_{3-} - \left(\frac{t}{2}\right)^2 \mathbf{X}_{3+} \end{pmatrix}, \tag{6.2}$$

and the unity root matrix $\mathbf{A}_{50}$ (5.5) is now

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & M & (\tau/2)\mathbf{X}^{3-} \\ -M & 0 & (t/2)\mathbf{X}^{3-} \\ (-\tau/2)\mathbf{X}_{3+} & (-t/2)\mathbf{X}_{3+} & \mathbf{A}_{30} \end{pmatrix}.$$

The scale factor $c^2(1+\gamma^2)/\gamma^2$ on $\mathbf{X}_{5-}$ (6.2) arises as a result of methods used in [3]. Its physical significance, if any, is not currently clear, albeit it does give the URM5 conservation equation $\mathbf{X}^{5-}\mathbf{X}_{5-}$ a fourth degree term in the quadratic velocities, C^2 and M^2; see [3], Section (7). However, given $\mathbf{X}_{5-}$ is an eigenvector, and hence arbitrary to within a scale factor, it will be safely disregarded for now. Note that it could equally well be transferred on to the $\mathbf{X}^{5-}$ or $\mathbf{X}_{5+}$ eigenvectors, whilst keeping the same, invariant inner products. There are also two complex eigenvectors in this mass solution, $\mathbf{X}_{5i+}$ and $\mathbf{X}_{5i-}$ [3], but these are not shown or required here as unnecessary.

6.2 Newton II

The URM3 component of $\mathbf{X}_{5-}$ (6.2) can be converted to a Newtonian form using the evolutionary form of the $\mathbf{X}_{3-}$ eigenvector, reproduced below from Section (2.13),

$$\mathbf{X}_{3-} = -t_3^2\mathbf{X}_{3+} + 2t_3\mathbf{X}'_{30} + \mathbf{X}'_{3-} \text{ (2.52)},$$

$$\mathbf{X}'_{30} = \mathbf{X}_{30}(t_3 = 0)\,,\ \mathbf{X}'_{3-} = \mathbf{X}_{3-}(t_3 = 0)\,,\text{ (2.51)},$$

Beforehand, however, the URM3 temporal (evolutionary) parameter t_3 is related to laboratory time t as per t_4 (6.1), i.e.

$$t_3 = -\frac{t}{2}.$$

By substituting this into $\mathbf{X}_{3-}$ (2.52), and then substituting $\mathbf{X}_{3-}$ into $\mathbf{X}_{5-}$ (6.2), the position vector $\mathbf{X}_{5-}$, without any scale factor, becomes

$$\mathbf{X}_{5-} = \begin{pmatrix} 0 \\ -ct \\ -\dfrac{t^2}{2}\mathbf{X}_{3+} + t\mathbf{X}'_{30} + \mathbf{X}'_{3-} \end{pmatrix},\ c,t \text{ form.} \tag{6.3}$$

Given $\mathbf{X}_{3+}$ represents a constant, negative acceleration, Section (2.15), with initial velocity $\mathbf{X}'_{30}$ and initial position $\mathbf{X}'_{3-}$, then the URM3 component of $\mathbf{X}_{5-}$ (6.3) can now be expressed as the position vector $\mathbf{r}$ of an object at time t, subject to a constant acceleration $\mathbf{a}$, with initial velocity $\mathbf{v}'$, initial position $\mathbf{r}'$, as in:

$$\mathbf{r} = -\frac{t^2}{2}\mathbf{X}_{3+} + t\mathbf{X}'_{30} + \mathbf{X}'_{3-} \ = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}'t + \mathbf{r}' .$$

$\mathbf{a} = -\mathbf{X}_{3+}$, constant acceleration,

$\mathbf{v}' = \mathbf{X}'_{30}$, initial velocity,

$\mathbf{r}' = \mathbf{X}'_{3-}$, initial position.

This is, of course, just Newton II, $\mathbf{F} = m\mathbf{a}$, for constant acceleration $\mathbf{a}$. Hence the URM5 mass solution is returned to a Newtonian form, and thereby completes this paper.

7 Summary

URMT starts with three linear, Diophantine equations in three unknowns, where each unknown is coupled to the other two by three dynamical variables and their three conjugates - all six dynamical variables defined as unity roots (or primitive roots). From this abstract starting point, and by application of an equally abstract invariance principle, a consistent physical interpretation of the resulting eigenvector solution is obtained. Notably, under the standard physical interpretation, Section (2.15), the eigenvectors can be associated with acceleration, velocity and position vectors that evolve with time, where time itself is a variational parameter.

The eigenvector inner products are conservation equations and, for a unity eigenvalue, give the three most basic scalars {0,1,2}, all of which are invariant to temporal evolution or other variations in the dynamical variables and eigenvectors. The characteristic, eigenvalue equation of the unity root matrix can also be consistently equated with an energy conservation equation and, in particular, upon extension to higher dimensions is seen to be the relativistic energy-momentum equation, with the invariant eigenvalue none other than the speed of light. Furthermore, temporal evolution of the higher dimensions leads to the geometric property of compactification, where the higher dimensions appear to shrink relative to the lower dimensions over long evolutionary periods.

An example, five-dimensional 'Doppler solution' shows a huge initial acceleration at the earliest instance of time, decaying in accordance with the Hubble expansion law. A second example shows how relativistic mass can be introduced implicitly into URMT by addition of a single, new dynamical variable, i.e. the 'reduced velocity'. This leads directly to the relativistic energy-momentum equation and a reformulation of the rest mass energy in a symmetric form involving both the reduced velocity and the speed of light. Lastly, such a solution is reduced to a Newtonian form in its first three dimensions.

8 Conclusion

What originated as a study in congruence relations and linear Diophantine equations, in particular with regard to their symmetry and invariants, appears to generate a surprisingly rich field of physical phenomena, and supports the author's premise that, at the smallest, Planck scale, nature reduces to some very simple rules, with its laws formulated as integer equations, and more the realm of number theory than physics. The laws of nature are thus reduced to the legal combinations of integers, which currently appear to be of quadratic degree, in particular hyperbolic and n-dimensional Pythagoras.

9 References

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